Finite condensate phase diffusion with phonon-number relaxation in a concave-phonon kinetic model

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Abstract

A superfluid's phase spreads because its phonon occupations fluctuate. For a concave acoustic branch, collisions that conserve phonon number can leave sufficiently slow modes to produce anomalous phase spreading. Number-changing collisions relax the phonon population, but a finite chemical equilibration rate does not by itself settle the phase problem: phase noise weights a continuum of arbitrarily soft modes. We prove that the energy-projected phase Green–Kubo functional is finite and strictly positive when the full leading isotropic two-to-two and two-to-three collision forms are both present with fixed positive coefficients. The proof retains gain terms and interference in the complete two-to-three amplitude. A regular family of collisions with a soft outgoing phonon transfers the entire infrared phase source to a residual controlled by the two-to-two form. An explicit weighted coercivity estimate then bounds the inverse collision operator in the required source direction. Consequently the thermodynamically scaled phase variance grows asymptotically linearly in time in this kinetic model. Phonon number is allowed to fluctuate and relax; only the true energy invariant is projected out. The result gives neither a numerical diffusion coefficient nor a crossover time, and does not establish an all-time, fixed-temperature law for the microscopic superfluid.

1 Introduction

Temporal coherence asks how long a condensate retains a predictable phase. Even in an isolated system, fluctuations of the quasiparticle occupations make the phase velocity fluctuate. Long-lived modes can accumulate phase noise over time. The important issue is therefore not simply whether collisions occur, but whether they damp the modes that the phase actually probes strongly enough to make the time integral of its correlation function finite.

For a three-dimensional superfluid with a concave acoustic branch, the leading low-temperature phonon kinetics provides a particularly sharp version of this question. The two-to-two Landau–Khalatnikov collision process conserves both energy and phonon number. In its restricted kinetic regime, Castin obtained anomalous phase spreading, with a leading $t^{5/3}$ law at zero phonon chemical potential; the leading coefficient was obtained analytically in a rate approximation and supported by numerical treatment of the full operator [1]. At longer kinetic scales, two-to-three and inverse collisions change phonon number. Castin and Tsimokha supplied their coherent low-temperature amplitude and studied phonon-number relaxation [2].

The latter relaxation result is not already an answer about phase diffusion. A number-relaxation calculation concerns an integrated population imbalance. Phase diffusion concerns a different source vector and requires control of the inverse collision operator down to arbitrarily small phonon energies. A finite relaxation rate for one collective quantity does not exclude a divergent phase-weighted infrared contribution. Conversely, a single-mode damping rate does not control the full inverse operator when gain terms couple different energies.

Here we settle this phase-weighted question in the combined leading kinetic model. For each fixed finite positive coefficient of both collision processes, its phase Green–Kubo functional is

finite and positive. Thus number-changing collisions, together with the redistribution supplied by two-to-two collisions, are sufficient to give ordinary asymptotic linear phase-variance growth in this model. The proof covers the continuum down to zero energy without an infrared cutoff. It does not assume a global spectral gap.

The central mechanism has two parts. First, collisions with an arbitrarily soft *outgoing* phonon retain nonzero coherent amplitude on a regular angular patch. A finite-norm current on those events represents the entire soft phase forcing and transfers its remaining effect to energies bounded away from zero. Second, the complete two-to-two event form controls this residual after its two collision-invariant moments are matched. The moment matching is a device inside the proof: it does not fix phonon number in the physical ensemble. Sections 2–4 specify the model and the local event geometry; Sections 5–8 give the source transfer, the coercivity estimate and the phase-variance conclusion.

2 Model, source and precise claim

2.1 Physical and limiting scope

We use the leading three-dimensional, isotropic, low-temperature collision forms for a concave phonon branch [1, 2]. The concavity parameter is strictly negative and nonzero. The small-angle and thermal-wave-number scalings that define these forms are retained. Their equation-of-state and phase-observable prefactors are finite and nonzero; degenerate cases in which a leading collision coefficient or the leading phase source vanishes are excluded. All fixed positive overall collision coefficients are allowed below. This does not assert estimates uniform near a vanishing coefficient or vanishing curvature.

The physical setting fixes atomic particle number and energy and takes the isotropic zero-total-momentum sector. Phonon number is not fixed. Its equilibrium chemical potential is zero. The reduced energy is $x = \hbar c q/(k_B T) > 0$, with Bose occupation and fluctuation measure

$$n(x) = \frac{1}{e^x - 1}, \qquad \mathrm{d}\mu(x) = m(x)\,\mathrm{d}x, \qquad m(x) = x^2 n(x)[1 + n(x)] = \left[\frac{x}{2\sinh(x/2)}\right]^2. \tag{1}$$

We work in $\mathcal{H} = L^2((0,\infty), \mathrm{d}\mu)$ and use occupation-potential variables $\delta n = n(1+n)g$.

After removal of the energy-conserved part and of a nonzero scalar physical prefactor, the phase source is

$$v(x) = x^3 - C_E x, \qquad C_E = \frac{\int_0^\infty x^6 n(x)[1+n(x)]\,\mathrm{d}x}{\int_0^\infty x^4 n(x)[1+n(x)]\,\mathrm{d}x} = \frac{20\pi^2}{7}. \tag{2}$$

Indeed, $n(1+n) = \sum_{j\geq 1} j e^{-jx}$ gives $\int_0^\infty x^p n(1+n)\,\mathrm{d}x = p!\zeta(p)$ for $p > 1$. Thus $v \in \mathcal{H}$, $v \neq 0$ and $\langle v, x\rangle_{\mathcal{H}} = 0$. The linear term $-C_E x$ at the infrared endpoint is retained. In particular, we do *not* subtract a constant to enforce $\langle v, 1\rangle_{\mathcal{H}} = 0$.

The order of interpretation matters. A phase is defined first at finite volume. At fixed density, one then takes the thermodynamic covariance with the particle-number factor N retained, and finally takes the long-time spectral limit of the leading kinetic model. The result below concerns this N-scaled variance. It is not obtained by first allowing an unscaled variance to vanish in the thermodynamic limit.

2.2 Collision forms and domains

Let $\mathcal{E}_{22}$ and $\mathcal{E}_{23}$ be the nonnegative event forms specified in Sections 3 and 4. For fixed finite $a, b > 0$, write

$$\mathcal{E}[g] = a\mathcal{E}_{22}[g] + b\mathcal{E}_{23}[g], \qquad \mathcal{D} = \mathcal{D}(\mathcal{E}_{22}) \cap \mathcal{D}(\mathcal{E}_{23}), \tag{3}$$

and let L be the associated nonnegative self-adjoint operator. The model uses densely defined closed event forms and their common closed sum, with the exact energy invariant x included in the domain. These domain conditions are part of the kinetic realization of the model, not a claim about the domain of a microscopic many-body generator.

For clarity, the standard event construction underlying these conditions can be described directly. For an event measure W and collision difference Tg, the maximal realization is $\{g \in \mathcal{H} : Tg \in L^2(W)\}$, with form $\|Tg\|^2_{L^2(W)}$. Absolute continuity of each leg marginal with respect to μ makes this operator closed: an $\mathcal{H}$-convergent sequence has a subsequence converging almost everywhere at every event leg, which identifies the $L^2(W)$ limit of its collision differences. The finite total equilibrium event mass in the leading kernels makes bounded functions finite-energy; their density supplies a dense domain. In the two-to-three case, this is the convergent equilibrium event integral underlying the number-relaxation calculation [2]; in the two-to-two case it follows also from the explicit kernel below. One may instead take the Friedrichs closure of the natural event-form domain containing the energy invariant. It is a restriction of the maximal realization. Our inequality holds on the maximal event domain and hence on either such realization. We require no unproved equality of minimal and maximal domains, nor an assertion that a Hilbert-dense collection is automatically a form core.

The notation

$$F(0) = \left\langle v, L^{-1}v \right\rangle := \sup_{g\in\mathcal{D}} \{2\,\mathrm{Re}\,\langle v, g\rangle_{\mathcal{H}} - \mathcal{E}[g]\} \tag{4}$$

means the extended spectral quadratic form of the inverse on the energy-orthogonal subspace. Finiteness means $v \in \mathrm{Dom}(L^{-1/2})$; it need not mean $v \in \mathrm{Dom}(L^{-1})$.

Theorem 1. *For the collision forms and source above, and every fixed finite $a, b > 0$,*

$$0 < F(0) < \infty. \tag{5}$$

The null space of the combined form is exactly $\mathrm{span}\{x\}$. *In the normalization in which v is the phase source, the limiting scaled kinetic phase variance satisfies*

$$\lim_{t\to\infty} \frac{\mathcal{V}(t)}{t} = 2F(0) > 0. \tag{6}$$

Nonzero scalar physical phase prefactors multiply the variance by their square.

The proof will give the explicit source-specific bound

$$|\langle v, g\rangle_{\mathcal{H}}|^2 \le \left(\frac{D}{aC_\epsilon} + \frac{K}{b}\right)\mathcal{E}[g], \qquad g \in \mathcal{D}, \qquad C_\epsilon = \frac{\pi\epsilon^7 e^{-\epsilon}}{108}, \tag{7}$$

where $\epsilon = 10^{-6}/4$, and the finite constants D and K are constructed below. Their loose bounds are intended to prove finiteness, not to estimate the physical diffusion coefficient.

3 Complete leading collision kernels

3.1 Two-to-two events and normalization

Put $k' = q + q' - k$ and

$$T_{22}g = g(q) + g(q') - g(k) - g(k'), \tag{8}$$

$$\begin{aligned} B_{22} &= n(q)n(q')[1 + n(k)][1 + n(k')] \\ &= \frac{e^{-(q+q')}}{(1 - e^{-q})(1 - e^{-q'})(1 - e^{-k})(1 - e^{-k'})}. \end{aligned} \tag{9}$$

The full isotropic leading event form is

$$\mathcal{E}_{22}[g] = \frac{\pi}{3}\int_0^\infty \mathrm{d}q \int_0^\infty \mathrm{d}q' \int_0^{q+q'} \mathrm{d}k\; qq'kk' \min(q,q',k,k')^3 B_{22}|T_{22}g|^2. \tag{10}$$

This includes every gain term through the collision difference. Its normalization follows from the reduced linearized operator, equation (034) of Ref. [1]. In occupation-potential variables the linearized bracket is $B_{22}T_{22}g$, with radial coefficient $(4\pi/3)q'kk'\min(q,q',k,k')^3/q$. Multiplication by $q^2\overline{g(q)}$ and integration gives an event integral with prefactor $4\pi/3$. Incoming exchange, outgoing exchange, and reversal preserve the measure. Averaging the four tagged terms gives $|T_{22}g|^2/4$, yielding (10). The radial-conservation Jacobian is one. Four diagonal tags restore the source coefficient $4\pi/3$.

As a normalization check, the source diagonal rate has leading behavior $d_4(x) \sim (16\pi^5/135)x^3$ at zero chemical potential [1]. We do not use this rate as a replacement for the operator. Both 1 and x are invariants of (10).

3.2 Two-to-three geometry and the coherent amplitude

Use incoming radii $q_1, q_2 > 0$ and outgoing radii $k_1, k_2, k_3 > 0$ with

$$Q = q_1 + q_2 = k_1 + k_2 + k_3, \qquad k_3 = Q - k_1 - k_2. \tag{11}$$

The angular variables are two-dimensional transverse vectors. Set the incoming directions to $\boldsymbol{\beta}_1 = (\theta, 0)$ and $\boldsymbol{\beta}_2 = 0$, where $\theta \geq 0$. Let $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ be independent outgoing angles and set

$$\boldsymbol{\alpha}_3 = \frac{q_1(\theta,0) - k_1\boldsymbol{\alpha}_1 - k_2\boldsymbol{\alpha}_2}{k_3}. \tag{12}$$

These are the reduced small-angle variables, with their common thermal and curvature scale already removed. Define

$$C = \frac{q_1^3 + q_2^3 - k_1^3 - k_2^3 - k_3^3}{4}, \tag{13}$$

$$W = q_1\theta^2 - \sum_{i=1}^{3} k_i|\boldsymbol{\alpha}_i|^2 - C. \tag{14}$$

The reduced energy shell is $W = 0$. The shift from the centered angular coordinates of Ref. [2] to this incoming-direction gauge has unit Jacobian.

A convenient rationalized factor is

$$R(\ell, \boldsymbol{z}) = \frac{\ell^2}{|\boldsymbol{z}|^2 + 3\ell^2/4}. \tag{15}$$

Write

$$\begin{aligned} H &= R(Q, (\theta, 0)), \\ O_1 &= R(k_2 + k_3, \boldsymbol{\alpha}_2 - \boldsymbol{\alpha}_3), \\ O_2 &= R(k_1 + k_3, \boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_3), \\ O_3 &= R(k_1 + k_2, \boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2), \\ M_{\alpha i} &= R(q_\alpha - k_i, \boldsymbol{\beta}_\alpha - \boldsymbol{\alpha}_i), \quad \alpha = 1, 2, \quad i = 1, 2, 3. \end{aligned} \tag{16}$$

There are ten primitive denominators. In this notation the complete reduced amplitude, with its fifteen terms displayed, is

$$\begin{aligned}\overline{\mathcal{A}} = {} & k_1 HO_1 + k_2 HO_2 + k_3 HO_3 \\ & - q_2 M_{11} O_1 - q_2 M_{12} O_2 - q_2 M_{13} O_3 \\ & - q_1 M_{21} O_1 - q_1 M_{22} O_2 - q_1 M_{23} O_3 \\ & + k_3 M_{11} M_{22} + k_2 M_{11} M_{23} + k_1 M_{12} M_{23} \\ & + k_3 M_{21} M_{12} + k_2 M_{21} M_{13} + k_1 M_{22} M_{13}. \end{aligned} \tag{17}$$

This is the coherent sum in the reduced amplitude of Ref. [2]. Intermediate diagram-ordering Heaviside factors have already been combined in that expression. It is $|\overline{\mathcal{A}}|^2$, not the sum of the squares of these terms, that enters the form. Rationalization removes the apparent division by $q_\alpha - k_i$ wherever the denominator in (15) is positive. No assertion of a regular global extension at simultaneous zero numerator and zero denominator is needed here.

With

$$T_{23} g = g(q_1) + g(q_2) - g(k_1) - g(k_2) - g(k_3), \tag{18}$$

$$B_{23} = n(q_1) n(q_2) \prod_{i=1}^{3} [1 + n(k_i)], \tag{19}$$

the form is

$$\mathcal{E}_{23}[g] = \int |T_{23} g|^2 \, \mathrm{d}W_{23}, \tag{20}$$

where the positive event measure in these coordinates is

$$\mathrm{d}W_{23} = \frac{1}{12} \frac{q_1 q_2 k_1 k_2}{k_3} B_{23} |\overline{\mathcal{A}}|^2 \delta(W) \; \mathrm{d}q_1 \, \mathrm{d}q_2 \, \mathrm{d}k_1 \, \mathrm{d}k_2 \, \theta \, \mathrm{d}\theta \; \mathrm{d}^2\boldsymbol{\alpha}_1 \, \mathrm{d}^2\boldsymbol{\alpha}_2. \tag{21}$$

The radial domain is $q_1, q_2, k_1, k_2 > 0$, $k_1 + k_2 < Q$; the independent angular vectors range over $\mathbb{R}^2$. The symbol $\delta(W)$ in (21) is the Dirac shell delta. The factor $1/12 = 1/(2!3!)$ is the event symmetry normalization. Equation (21) is the exact measure of the *leading reduced* model, not the pre-limit microscopic collision measure. All omitted dimensional scalar factors are in b.

Energy conservation gives $T_{23} x = 0$, whereas $T_{23} 1 = -1$. Thus phonon number is relaxed by this form. The identities also explain why matching a number moment and projecting out an invariant are different operations.

4 A regular outgoing-soft event patch

Only a positive subset of (21) is needed. Retaining this subset for an estimate does not remove other events from the operator.

4.1 Exact coordinates and shell root

Set $\delta = 10^{-6}$. For every $0 < x < \delta$, choose

$$q_1 = a_0, \quad q_2 = b_0, \quad k_1 = c_0, \quad k_3 = x, \quad k_2 = d_0 = a_0 + b_0 - c_0 - x. \tag{22}$$

The seven free variables $y = (a_0, b_0, c_0, \theta, u, U_1, U_2)$ lie in the box Y defined by

$$\begin{aligned} & a_0, b_0, c_0, \theta \in [1 - \delta, 1 + \delta], \\ & \qquad u \in [1/2 - \delta, 1/2 + \delta], \qquad U_1, U_2 \in [-\delta, \delta]. \end{aligned} \tag{23}$$

Its volume is $V_Y = (2\delta)^7$. Set

$$\boldsymbol{\alpha}_1 = (u, z), \qquad \boldsymbol{\alpha}_3 = U = (U_1, U_2), \qquad \boldsymbol{\alpha}_2 = \frac{a_0(\theta, 0) - c_0(u, z) - xU}{d_0}. \tag{24}$$

The symbol z here denotes a shell coordinate, not fugacity. In these variables $W = D_0 - A_0 z^2 - B_0 z$, where

$$\begin{aligned} A_0 &= c_0 + c_0^2/d_0, \qquad B_0 = 2c_0 x U_2/d_0, \\ D_0 &= a_0\theta^2 - c_0 u^2 - \frac{(a_0\theta - c_0 u - xU_1)^2}{d_0} - x(U_1^2 + U_2^2) - \frac{x^2 U_2^2}{d_0} - C. \end{aligned} \tag{25}$$

Choose the positive branch

$$z = \frac{-B_0 + \sqrt{B_0^2 + 4A_0 D_0}}{2A_0}. \tag{26}$$

It is an exact root, rather than an approximate solution of the shell equation.

The radial Jacobian from (q_1, q_2, k_1, k_2) to (a_0, b_0, c_0, x) has absolute value one. At fixed $\boldsymbol{\alpha}_1$, the angular Jacobian from $\boldsymbol{\alpha}_2$ to U is $(x/d_0)^2$. Combining it with the radial prefactor $a_0 b_0 c_0 d_0/x$ in (21) gives $a_0 b_0 c_0 x/d_0$. The remaining angular factor is the original $\theta\,\mathrm{d}\theta$, without a replacement by a Cartesian rotation quotient.

4.2 Uniform interval bounds and their reproduction

Evaluating (22)–(26), (16) and (17) by outward rational interval arithmetic on the whole box, including $0 \le x \le \delta$, gives the following deliberately widened rational enclosures:

$$\begin{aligned} 0.4999904 < z &< 0.5000090, \\ 1.999955 < |\partial_z W| &< 2.000044, \\ -0.304856 < \overline{\mathcal{A}} &< -0.304666. \end{aligned} \tag{27}$$

Every terminating decimal in this paragraph and Table 1 denotes an exact rational number. In particular,

$$\tfrac{1}{4} \le |\overline{\mathcal{A}}| \le 1, \qquad 1 < |\partial_z W| < 3, \qquad \tfrac{1}{2} \le a_0, b_0, c_0, d_0, \theta \le 2. \tag{28}$$

The ten primitive denominators are strictly positive throughout the box.

Factor	Denominator enclosure	Factors sharing the enclosure
H	$[3.9999, 4.0001]$	H
O_i	$[1.2499, 1.2501]$	O_1, O_2
O_3	$[3.9999, 4.0001]$	O_3
$M_{\alpha i}$	$[0.4999, 0.5001]$	$M_{11}, M_{12}, M_{21}, M_{22}$
M_{13}	$[1.7499, 1.7501]$	M_{13}
M_{23}	$[0.7499, 0.7501]$	M_{23}

Table 1: Widened rational enclosures of $|\boldsymbol{z}|^2 + 3\ell^2/4$ in the ten factors (16). They hold on the continuous patch, not merely at sample points.

For reproducibility, the interval procedure is fully specified as follows. Use the exact endpoints in (23) and $x \in [0, 10^{-6}]$. After every arithmetic operation round the lower endpoint down and the upper endpoint up to the grid $10^{-24}\mathbb{Z}$. For a square use zero as the lower endpoint when the interval contains zero; otherwise take the smaller endpoint square. For a reciprocal, require that the interval exclude zero. For a square root of $[l, h]$, with $G = 10^{24}$, use

$$\left[\frac{\lfloor\sqrt{\lfloor lG^2\rfloor}\rfloor}{G}, \frac{\lfloor\sqrt{\lfloor hG^2\rfloor}\rfloor + 1}{G}\right]. \tag{29}$$

Compute d_0, then A_0, B_0, C, D_0, z in the displayed order, then the vectors in (24), the ten factors and the fifteen amplitude terms. The shell derivative is $|2A_0 z + B_0|$. These finite rational operations verify (27) and Table 1; the supplement supplies both an implementation and a separate check. The method encloses every point of the parameter box. It uses neither a finite sampling grid nor floating-point acceptance tests.

At the central limiting point $x = 0$, $a_0 = b_0 = c_0 = d_0 = \theta = 1$, $u = z = 1/2$ and $U = 0$, one has $\boldsymbol{\alpha}_2 = (1/2, -1/2)$ and $\overline{\mathcal{A}} = -32/105$. This point explains the choice of patch; the interval bounds establish its continuation to a set of positive volume for all $0 < x < \delta$.

4.3 Two-sided density bounds and hard-leg marginals

After resolving the shell by (26), the restricted event measure is

$$\mathrm{d}W_P = w(x, y)\,\mathrm{d}x\,\mathrm{d}y, \qquad w = \frac{1}{12}\frac{a_0 b_0 c_0}{d_0}\,\frac{x B_{23}\theta|\overline{\mathcal{A}}|^2}{|\partial_z W|}. \tag{30}$$

On the patch, $a_0 b_0 c_0/d_0 \geq 1/16$. For $0 < h \leq 2$, $n(h) > 1/8$, since $e < 3$. The two incoming Bose factors therefore give at least $1/64$; the two hard stimulated factors are at least one; and $1 + n(x) \geq 1/x$. With (28), this yields

$$w(x, y) \geq w_0 := \frac{1}{12 \cdot 16 \cdot 3 \cdot 2 \cdot 16 \cdot 64} = \frac{1}{1179648}. \tag{31}$$

For an upper bound, $h \geq 1/2$ gives $n(h) \leq 2$ and $1 + n(h) \leq 3$. Also $x[1 + n(x)] \leq e^x < 3$. Hence $xB_{23} \leq 2^2 \cdot 3^2 \cdot 3 = 108$. Using $a_0 b_0 c_0/d_0 \leq 16$, $\theta \leq 2$, $|\overline{\mathcal{A}}|^2 \leq 1$ and $|\partial_z W|^{-1} \leq 1$ gives

$$w(x, y) \leq w_1 := 288. \tag{32}$$

Consequently, the tagged marginal

$$\rho(x) = \int_Y w(x, y)\,\mathrm{d}y \quad \text{satisfies} \quad w_0 V_Y \leq \rho(x) \leq w_1 V_Y, \qquad 0 < x < \delta. \tag{33}$$

The soft radial factor x has been compensated by its stimulated Bose factor. These are bounds on the full coherent event measure.

Every hard-coordinate pushforward of a bounded current on this patch has a bounded Lebesgue density supported in $[1/2, 2]$. For a_0, b_0, c_0 this follows by integrating over the remaining bounded box variables. For d_0 use $a_0 \mapsto d_0 = a_0 + b_0 - c_0 - x$, whose Jacobian is one, and the same bound. This local statement is all that will be needed; it does not presuppose global regularity of a two-to-three gain kernel.

A nonzero tagged outgoing density is consistent with the source's distinction between incoming- and outgoing-soft limits [2]. Its incoming fixed-hard asymptotic has coherent soft suppression, while the outgoing limit uses a different angular rescaling. Transferring the former suppression to the latter would miss the patch above. The patch alone, however, is not a bound on the full inverse operator: gain terms can still correlate hard and soft components. The next two sections address that issue.

5 Finite-cost transfer of the phase source

Let $\epsilon = \delta/4$ and define the Lebesgue source density

$$f(x) = v(x)m(x)\mathbf{1}_{(0,\epsilon)}(x). \tag{34}$$

On the patch set

$$J_0(x, y) = -\frac{f(x)}{\rho(x)}, \tag{35}$$

and set it to zero on all other events. The negative sign is required because x is an outgoing leg:

$$T_{23}g = g(a_0) + g(b_0) - g(c_0) - g(d_0) - g(x). \tag{36}$$

Thus $\langle J_0, T_{23}g\rangle_{L^2(W_{23})}$ equals the source pairing on $(0,\epsilon)$ plus a forcing supported at hard energies. The following calculations can first be read for real functions; conjugation gives the complex Hilbert-space identities as well.

Since $m \le 1$ and $|v(x)| \le (C_E + \delta^2)x$ on this interval,

$$K_0 := \|J_0\|^2_{L^2(W_{23})} = \int_0^\epsilon \frac{|f(x)|^2}{\rho(x)}\,\mathrm{d}x \le \frac{(C_E+\delta^2)^2\epsilon^3}{3w_0V_Y} < \infty. \tag{37}$$

In particular J_0 is bounded, including as $x \downarrow 0$. Its hard pushforward density is bounded by the preceding section.

The remaining source must have zero moments against both two-to-two invariants before the two-to-two comparison can control it. Only its energy moment already has this property. To match the number moment, put

$$M_0 = \langle v, 1\rangle_{\mathcal{H}} - \int_0^\epsilon f(x)\,\mathrm{d}x, \qquad A = \{(x,y) \in P : \delta/2 < x < 3\delta/4\}. \tag{38}$$

Its event mass $W_A = \int_A \mathrm{d}W_{23}$ satisfies $W_A \ge w_0V_Y\delta/4 > 0$. Set

$$J_1 = -\frac{M_0}{W_A}\mathbf{1}_A, \qquad K_1 := \|J_1\|^2_{L^2(W_{23})} = \frac{M_0^2}{W_A} \le \frac{4M_0^2}{w_0V_Y\delta} < \infty. \tag{39}$$

Because $T_{23}1 = -1$, this current contributes exactly M_0 to the number moment. Because $T_{23}x = 0$, it contributes nothing to the energy moment. The supports of J_0 and J_1 in the tagged variable are disjoint, so

$$J = J_0 + J_1, \qquad \|J\|^2_{L^2(W_{23})} = K_0 + K_1 =: K < \infty. \tag{40}$$

Define a residual through its signed leg pushforward density:

$$r\,\mathrm{d}\mu = v\,\mathrm{d}\mu - T_{23}^*J. \tag{41}$$

Here T_{23}^*J is initially a signed measure: each event contributes its current with the incoming signs and the outgoing signs of (18). The density conclusions already proved make (41) an ordinary function identity almost everywhere. Its essential properties are

$$r = 0 \quad (0 < x < \epsilon), \qquad \langle r, 1\rangle_{\mathcal{H}} = 0, \qquad \langle r, x\rangle_{\mathcal{H}} = 0. \tag{42}$$

Indeed, the outgoing contribution of J_0 cancels v below ϵ, and neither its hard legs nor any leg of J_1 lies there. For the number moment, $\langle T_{23}^*J_0, 1\rangle = \int_0^\epsilon f(x)\,\mathrm{d}x$ and $\langle T_{23}^*J_1, 1\rangle = M_0$. For the energy moment both currents give zero event by event, and (2) gives the remaining zero. No number-orthogonality of v has been assumed.

Write $s(x) = r(x)m(x)$. On compact intervals above ϵ it is bounded, since its hard and annular pushforward densities are bounded. For $x > 2$ it is exactly $v(x)m(x)$, since every modified leg lies below two. Therefore

$$D := \int_\epsilon^\infty |s(x)|^2 e^x\,\mathrm{d}x < \infty. \tag{43}$$

The tail is explicit:

$$m(x)^2e^x = \frac{x^4e^{-x}}{(1-e^{-x})^4}. \tag{44}$$

Multiplication by the square of the cubic polynomial v leaves a polynomial times an exponentially decaying function. The compact part of (43) has positive lower endpoint and bounded density. These observations also prove $r \in \mathcal{H}$ and absolute convergence of both moments in (42).

All individual leg integrals defining (41) are absolutely convergent for every $g \in \mathcal{H}$. On the soft interval this is Hilbert-space Cauchy–Schwarz for v and g. On hard and annular compact intervals, m is bounded below by a positive number, so local L^2 and L^1 bounds suffice. Accordingly,

$$\langle v, g\rangle_{\mathcal{H}} = \langle r, g\rangle_{\mathcal{H}} + \langle J, T_{23} g\rangle_{L^2(W_{23})}, \qquad g \in \mathcal{D}, \tag{45}$$

holds on the full form domain, without a separate form-core approximation argument.

6 A weighted hard-energy estimate for the full two-to-two form

Restrict (10) to events with all four radii at least ϵ, and call this positive restriction $\mathcal{E}_{22}^{\mathrm{hard}}$. Put

$$q = \epsilon + y, \quad q' = \epsilon + y', \quad k = \epsilon + z, \quad k' = \epsilon + y + y' - z, \qquad h(y) = g(\epsilon + y). \tag{46}$$

Then $y, y' > 0$ and $0 < z < y + y'$. The radial product times $\min^3$ is at least ϵ^7, and $B_{22} \geq e^{-(q+q')}$. Consequently

$$\mathcal{E}_{22}^{\mathrm{hard}}[g] \geq \frac{\pi}{3}\epsilon^7 e^{-2\epsilon}\mathcal{E}_{\mathrm{c}}[h], \tag{47}$$

where

$$\mathcal{E}_{\mathrm{c}}[h] = \int_0^\infty \mathrm{d}y \int_0^\infty \mathrm{d}y'\, e^{-(y+y')} \int_0^{y+y'} \mathrm{d}z\, |h(y) + h(y') - h(z) - h(y + y' - z)|^2. \tag{48}$$

We prove the needed estimate for this entire redistribution form, including its gain terms.

6.1 A conditional redistribution operator

Divide the event measure in (48) by $s_0 = y + y'$ and denote the resulting form by $\mathcal{E}_0$. Its one-leg marginal is exactly $e^{-y}\,\mathrm{d}y$, since the z integral cancels the factor $1/s_0$ and $\int_0^\infty e^{-y'}\,\mathrm{d}y' = 1$. Hence the zero-total-energy endpoint creates no marginal singularity. Define

$$Ph(y) = \int_0^\infty \frac{e^{-y'}}{y + y'}\,\mathrm{d}y' \int_0^{y+y'} h(z)\,\mathrm{d}z. \tag{49}$$

This is a Markov operator, $P1 = 1$. Its weighted kernel is symmetric:

$$e^{-y}P(y, z) = \int_{\max(y,z)}^\infty \frac{e^{-s_0}}{s_0}\,\mathrm{d}s_0. \tag{50}$$

Thus P is self-adjoint in $L^2(e^{-y}\,\mathrm{d}y)$. Jensen's inequality and stationarity of this probability measure make it a bounded contraction.

For $n \geq 0$, direct integration yields

$$P(y^n) = \frac{1}{n+1}\sum_{j=0}^{n} \binom{n}{j} j!\, y^{n-j}. \tag{51}$$

In particular, P preserves every finite-degree polynomial space and acts on its leading coefficient by $1/(n+1)$.

6.2 Complete Laguerre diagonalization

The ordinary Laguerre polynomials

$$L_n(y) = \frac{e^y}{n!}\frac{\mathrm{d}^n}{\mathrm{d}y^n}(e^{-y}y^n) \tag{52}$$

are orthonormal in $L^2(e^{-y}\,\mathrm{d}y)$. To see the normalization and not just orthogonality, integration by parts n times shows orthogonality to all lower-degree polynomials and $\langle L_n, y^n\rangle = (-1)^n n!$. Since the leading coefficient of L_n is $(-1)^n/n!$, its squared norm is one. Boundary terms vanish both at zero and at infinity.

Completeness is also needed. If h is orthogonal to every polynomial, its transform

$$H_h(\zeta) = \int_0^\infty h(y) e^{-y} e^{\zeta y}\,\mathrm{d}y \tag{53}$$

is analytic for $\operatorname{Re}\zeta < 1/2$ by Cauchy–Schwarz. Every derivative at zero vanishes. Analytic uniqueness makes H_h zero on that half-plane. Restricting to the imaginary axis and using uniqueness of the Fourier transform of the L^1 function he^{-y} gives $h = 0$. The Laguerre polynomials therefore form a complete orthonormal basis, not merely a useful finite collection of test functions.

By self-adjointness, PL_n is orthogonal to every polynomial of smaller degree. Equation (51) fixes its leading term, hence

$$PL_n = \frac{1}{n+1} L_n. \tag{54}$$

Boundedness of P and completeness extend this diagonalization to all of $L^2(e^{-y}\,\mathrm{d}y)$.

The four-leg event symmetry gives the exact identity

$$\mathcal{E}_0[h] = 4\,\langle h, (I + \Pi_1 - 2P)h\rangle_{L^2(e^{-y}\,\mathrm{d}y)}\,, \tag{55}$$

where Π_1 projects onto constants. To verify it, the four square terms contribute $4\,\|h\|^2$. The two within-pair cross terms contribute $4|\,\langle h, 1\rangle\,|^2$; the four incoming-outgoing cross terms contribute $-8\,\langle h, Ph\rangle$. These statements first follow by Fubini for bounded compact functions. The event inequality $|\sum_{i=1}^4 \sigma_i h_i|^2 \le 4\sum_{i=1}^4 |h_i|^2$ and the one-leg marginals give $\mathcal{E}_0[h] \le 16\,\|h\|^2$, so (55) extends continuously to all L^2 functions.

The constant mode and $L_1 = 1 - y$ are exactly the two null modes. For $n \ge 2$, the eigenvalues of $I + \Pi_1 - 2P$ are $1 - 2/(n+1) \ge 1/3$. Let h_0 be h with its affine orthogonal projection removed. Then

$$\mathcal{E}_0[h] = \mathcal{E}_0[h_0] \ge \frac{4}{3}\,\|h_0\|^2_{L^2(e^{-y}\,\mathrm{d}y)}\,. \tag{56}$$

6.3 Removing small-total-energy events

To recover the undivided form $\mathcal{E}_{\mathrm{c}}$, remove from $\mathcal{E}_0$ the events with $s_0 < \eta$. Each of their one-leg marginals is bounded by $\eta e^{-y}\,\mathrm{d}y$, because $\int_0^{\eta-y} e^{-y'}\,\mathrm{d}y' \le \eta$ when $0 < y < \eta$. Consequently,

$$\mathcal{E}_0^{s_0<\eta}[h_0] \le 16\eta\,\|h_0\|^2\,. \tag{57}$$

Affine functions have zero event difference even after restriction of the event set. Taking $\eta = 1/24$ in (56)–(57) gives

$$\mathcal{E}_0^{s_0\ge\eta}[h] \ge \frac{2}{3}\,\|h_0\|^2\,. \tag{58}$$

The measure of $\mathcal{E}_{\mathrm{c}}$ is s_0 times that of $\mathcal{E}_0$, so

$$\mathcal{E}_{\mathrm{c}}[h] \ge \frac{1}{36}\,\|h_0\|^2_{L^2(e^{-y}\,\mathrm{d}y)}\,. \tag{59}$$

This remains valid when $\mathcal{E}_{\mathrm{c}}[h] = \infty$.

Combining (47) and (59) and shifting the exponential norm back to x yields

$$\mathcal{E}_{22}[g] \ge \mathcal{E}_{22}^{\mathrm{hard}}[g] \ge C_\epsilon \inf_{A,B\in\mathbb{C}} \int_\epsilon^\infty e^{-x}|g(x) - A - Bx|^2\,\mathrm{d}x, \qquad C_\epsilon = \frac{\pi\epsilon^7 e^{-\epsilon}}{108}. \tag{60}$$

The factor $e^{+\epsilon}$ from the change of norm is included in this constant. Every $g \in \mathcal{H}$ has the required hard-energy restriction in $L^2(e^{-x}\,\mathrm{d}x)$, since

$$m(x) = \frac{x^2 e^{-x}}{(1-e^{-x})^2} \geq \epsilon^2 e^{-x}, \qquad x \geq \epsilon. \tag{61}$$

Thus (60) applies to every $g \in \mathcal{D}$. It is a weighted estimate modulo affine functions on the hard interval; it is not a global gap in $\mathcal{H}$ and was not inferred from a finite matrix or a loss-only rate.

7 The source-dual bound and null space

Choose A, B minimizing the norm in (60). The moment identities (42) allow their subtraction in the residual pairing. Weighted Cauchy–Schwarz, (43) and (60) give

$$\begin{aligned} |\langle r, g\rangle_{\mathcal{H}}|^2 &= \left| \int_\epsilon^\infty s(x)[g(x) - A - Bx]\,\mathrm{d}x \right|^2 \\ &\leq D \int_\epsilon^\infty e^{-x} |g(x) - A - Bx|^2\,\mathrm{d}x \leq \frac{D}{C_\epsilon}\mathcal{E}_{22}[g]. \end{aligned} \tag{62}$$

For complex functions one inserts the conjugates consistent with the chosen inner-product convention; the estimate is identical. The current pairing obeys

$$|\langle J, T_{23} g\rangle_{L^2(W_{23})}|^2 \leq K \mathcal{E}_{23}[g]. \tag{63}$$

Using the dual identity (45) and applying the two-component weighted Cauchy–Schwarz inequality gives

$$\begin{aligned} |\langle v, g\rangle_{\mathcal{H}}|^2 &\leq \left(\sqrt{\frac{D}{C_\epsilon}\mathcal{E}_{22}[g]} + \sqrt{K\mathcal{E}_{23}[g]} \right)^2 \\ &\leq \left(\frac{D}{aC_\epsilon} + \frac{K}{b} \right) (a\mathcal{E}_{22}[g] + b\mathcal{E}_{23}[g]), \qquad g \in \mathcal{D}. \end{aligned} \tag{64}$$

There is no extra factor two. Define $C_v = D/(aC_\epsilon) + K/b$. This constant is finite for every fixed finite $a, b > 0$. Every infrared source contribution and the complete high-energy tail has been accounted for. Additional nonnegative collision events can only improve this upper bound for the fixed source and domain.

We can also identify the combined null space directly. If $\mathcal{E}[g] = 0$, (60) gives $g(x) = A + Bx$ almost everywhere for $x \geq \epsilon$. On the patch with $\epsilon < x < \delta$, all five legs are in this interval. Equation (36) becomes $T_{23} g = -A$. The positive event mass forces $A = 0$. On $0 < x < \epsilon$, the hard values are now B times their radii, so $T_{23} g = Bx - g(x)$. The lower marginal bound (33) forces $g(x) = Bx$ there also. Conversely x has zero event difference in both forms. Hence

$$\ker L = \mathrm{span}\{x\}. \tag{65}$$

The number mode has been relaxed by the actual collision form; it has not been projected away.

8 Finite inverse functional and linear phase variance

By (64), the variational expression (4) is bounded above by C_v: for $e = \mathcal{E}[g]$, $2\,\mathrm{Re}\,\langle v, g\rangle - e \leq 2\sqrt{C_v e} - e \leq C_v$. It is strictly positive. Indeed, density of $\mathcal{D}$ and $v \neq 0$ provide a finite-energy g with nonzero pairing. Its energy is positive by (64), and optimizing a scalar multiple of g gives a strictly positive value of the variational functional. This proves

$$0 < F(0) \leq \frac{D}{aC_\epsilon} + \frac{K}{b} < \infty. \tag{66}$$

Let σ be the positive spectral measure of L in the vector v. Since the sole invariant is energy and $v \perp x$, σ has no atom at zero. Equation (66) says

$$\int_{(0,\infty)} \frac{1}{\lambda}\, \mathrm{d}\sigma(\lambda) = F(0) < \infty. \tag{67}$$

The phase-velocity covariance in the chosen kinetic normalization is $\left\langle v, e^{-tL} v \right\rangle_{\mathcal{H}} = \int e^{-\lambda t}\, \mathrm{d}\sigma(\lambda)$. Integrating it twice gives

$$\frac{\mathcal{V}(t)}{t} = 2 \int_{(0,\infty)} \left[\frac{1}{\lambda} - \frac{1 - e^{-\lambda t}}{t\lambda^2} \right] \mathrm{d}\sigma(\lambda). \tag{68}$$

The bracket is nonnegative, is bounded by $1/\lambda$, and tends to $1/\lambda$ for every $\lambda > 0$. Dominated convergence proves

$$\frac{\mathcal{V}(t)}{t} \longrightarrow 2F(0), \qquad 0 < 2F(0) < \infty, \tag{69}$$

completing Theorem 1. This conclusion uses a source-specific integrable spectral weight, not a uniform relaxation time for all modes.

9 Interpretation and limitations

The physical consequence is that the combined leading collision model has a finite, nonzero phase diffusion coefficient. Number-changing events are strong enough, in the phase-weighted sense, to remove the infrared obstruction to ordinary asymptotic phase spreading. That statement requires more than finite chemical equilibration. The proof exhibits the missing connection: an outgoing-soft event current has finite cost while representing every sufficiently soft part of the phase source, and the remaining redistribution cost is controlled by the full two-to-two form.

The contrast with the restricted two-to-two theory is useful but must be drawn carefully. That theory conserves phonon number and uses a phase source with both invariant components removed; at zero phonon chemical potential it has the anomalous behavior discussed in Ref. [1]. In the present combined model, phonon number relaxes and the source has only its energy component removed. These are different collision-invariant structures and different source projections. Our theorem neither invalidates the earlier regime nor compares the same observable in the same fixed ensemble across an explicitly calculated crossover. It proves the ultimate spectral statement of the declared combined leading model. It gives no crossover time between experimentally or kinetically distinct regimes.

The distinction between incoming and outgoing soft phonons is also consequential. Coherent cancellation in an incoming fixed-hard limit cannot be imposed on every tagged leg. The regular outgoing patch retains a nonzero coherent amplitude, and its Bose enhancement compensates its small radial factor. This is why the patch can carry the soft forcing at finite cost. Yet treating that observation as a standalone damping-rate argument would still be insufficient: the hard forcing it creates must be controlled with all gain terms retained. Equation (60) provides precisely that control.

Several boundaries of the result remain explicit. The constants are extremely conservative and are not useful numerical estimates of $F(0)$. They are not uniform as either a or b tends to zero. We obtain no new anomalous exponent, no numerical diffusion coefficient and no quantitative approach-to-linearity rate. The parameter ϵ is an internal partition in the proof: J_0 represents the full interval $(0, \epsilon)$, so no arbitrary physical infrared cutoff has been inserted. Nor do we need a complete angular atlas or a global upper estimate on the two-to-three kernel; a positive regular subset suffices for this direction of inequality.

Finally, a leading kinetic theorem is not an all-time fixed-temperature statement about the real superfluid. Its microscopic interpretation needs a controlled, nonempty joint temperature,

time and volume window. Finite-volume spectral resolution, damping-width modifications, nonresonant terms, omitted vertices and other processes require their own estimates. For a fixed source and a compatible common domain, adding positive forms preserves our finite upper bound, but that mathematical monotonicity does not establish the missing microscopic limiting argument. The sequential scaled-variance interpretation stated in Section 2 is therefore essential.

10 Conclusion

In the isotropic leading concave-phonon model with both two-to-two and two-to-three collisions, the energy-projected phase Green–Kubo functional is finite and strictly positive for every fixed finite pair of positive collision coefficients. A coherent outgoing-soft patch transfers the infrared source to a residual controlled by an explicit two-to-two redistribution inequality. The resulting spectral bound proves asymptotically linear growth of the thermodynamically scaled phase variance while allowing phonon number to relax. This resolves the phase-weighted infrared question within that kinetic model; its numerical coefficient, crossover and controlled microscopic extension remain separate questions.

Reproducibility

All analytical definitions needed for the proof are displayed above, including the complete two-to-two kernel, the fifteen-term two-to-three amplitude, the energy shell, the patch coordinates and the interval rules. The accompanying reproducibility supplement contains `verify_patch.py`, `independent_patch_check.py`, `reference-results.json` and `README.txt`. The scripts verify the uniform local amplitude, shell-gradient and denominator bounds with exact rational interval arithmetic. They do not numerically approximate a collision spectrum or diffusion coefficient. The remaining claims follow from the source-transfer identities, the complete Laguerre argument and the spectral limit given in the article. The bibliographic versions below specify the leading kernels and their physical assumptions.

References

[1] Y. Castin, *Phase spreading and temporal coherence of a pair-condensed Fermi gas at low temperature*, arXiv:1807.10476v2 (2019; first posted 2018). https://arxiv.org/abs/1807.10476v2.

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