Long-wavelength stability and finite-wavelength instability of a coherent neutrino equilibrium

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Abstract

Can stability against homogeneous and sufficiently long-wavelength perturbations justify a coherent neutrino state as a stable asymptotic candidate? We give a counterexample in an idealized two-flavor kinetic model with fixed diagonal repopulation targets. At one explicitly specified positive, isotropic coherent relative equilibrium, the complete axial angular-continuum linearization has uniformly bounded evolution for all dimensionless wave numbers $|q| \leq 2^{-62}$. At the identical state, $q = 3/64$ has a real eigenvalue strictly between 0.0001148273 and 0.0003590276 in units of the electron-neutrino relaxation rate. The long-wavelength proof retains the neutral flavor-phase mode, establishes positive phase diffusion, and controls the complementary semigroup. The unstable eigenvalue is certified by continuous-angle resolvent integration, a zero-free determinant contour, and reconstruction of a nonzero continuum eigenfunction. Thus it is not inferred from an angular truncation. This is an existence result for a vacuum-free, single-energy phenomenological reservoir model, not a prediction for a supernova or merger trajectory. The very small sufficient stable interval and large norm bound are conservative proof constants, not a measured stability threshold or transient amplitude.

1 Introduction

Collisional repopulation can sustain nonzero neutrino flavor coherence, making the stability of the resulting coherent state a different question from the onset of an instability around a flavor-diagonal state. Homogeneous calculations are useful for identifying possible late-time states, but a spatially varying perturbation also streams between angular sectors. A reliable linear stability statement must specify the background, retained density components, angular space and range of wave numbers.

Froustey's homogeneous, isotropic relaxation model predicts asymptotic states with nonzero coherence and emphasizes the role of diagonal repopulation [1]. Its stated scope does not establish spatial stability of those states. Analyses of collisional onset about flavor-diagonal distributions [2], and asymptotic prescriptions for fast conversion [3], concern related but distinct questions. More generally, coherent-background flavor-wave linearization is already part of local-equilibrium theory [4]. Bhattacharyya and Dasgupta already exhibit a collisionless flavor-diagonal distribution with a stable homogeneous mode and a finite-wave-number instability band [5]. Spatial structure and the possibility of finite-wavelength instabilities are not new principles.

The contribution here is a concrete, quantitatively certified separation at one and the same state: full axial-continuum boundedness at zero and sufficiently small wave number, together with a positive continuum eigenvalue at a finite wave number. All eight trace, population and coherence perturbations of both species are retained. The proof combines finite-dimensional exact algebra with an analytic estimate for the infinite angular complement and validated continuous-angle integration. No assertion of a new general dispersion-relation method or of literature-wide priority is needed for this counterexample.

The model makes a deliberate phenomenological choice. Fixed isotropic diagonal reservoirs are maintained while streaming is introduced explicitly. In the homogeneous source model, heavy-lepton relaxation is an effective description that includes replenishment associated with advection [1]. Treating it as a fixed local reservoir in a transport equation is an extension of that description, not a derived spatial closure. The interaction strength chosen below is not identified with a published source trajectory. These restrictions are essential to the interpretation.

2 Model and exact relative equilibrium

Let $v \in [-1, 1]$ be the axial direction cosine, with $\langle f \rangle = \int_{-1}^{1} f(v)\, dv/2$. The two 2×2 Hermitian density matrices obey

$$(\partial_\tau + v\partial_\zeta)N_v = -\mathrm{i}[H_v, N_v] + \tfrac{1}{2}\{K, N^{\mathrm{cl}} - N_v\}, \tag{1}$$

$$(\partial_\tau + v\partial_\zeta)\bar{N}_v = -\mathrm{i}[H_v, \bar{N}_v] + \tfrac{1}{2}\{\bar{K}, \bar{N}^{\mathrm{cl}} - \bar{N}_v\}, \tag{2}$$

$$H_v = x\left\langle (1 - vv')(N_{v'} - \bar{N}_{v'})\right\rangle - \tfrac{1}{2}\Omega\sigma_z, \tag{3}$$

where the average in (3) is over v', $K = \mathrm{diag}(1, r)$ and $\bar{K} = \mathrm{diag}(c, r)$. Vacuum mixing is zero; a common diagonal matter potential has been removed by flavor rotation. Both species have the same self-interaction sign. The term involving Ω is the rotating-frame term. A stationary solution in this frame is a relative equilibrium in the unrotated frame.

Densities are expressed in units of a reference density N_0; time and position are $\tau = \kappa_e t$ and $\zeta = \kappa_e z/c_{\mathrm{light}}$. Thus $q = kc_{\mathrm{light}}/\kappa_e$ and a dimensionless eigenvalue σ corresponds to physical growth rate $s = \kappa_e\sigma$. We use the following exact idealized inputs:

$$r = \frac{41}{10900}, \quad c = \frac{211}{763}, \quad N^{\mathrm{cl}} = \mathrm{diag}\left(\frac{104}{10}, \frac{71}{10}\right), \quad \bar{N}^{\mathrm{cl}} = \mathrm{diag}\left(\frac{126}{10}, \frac{71}{10}\right). \tag{4}$$

The ratios and target densities are inspired by the rounded point-A entries in Ref. [1], with $N_0 = 10^{33}\,\mathrm{cm}^{-3}$. They are exact parameters here; no uncertainty interval for those source values is claimed.

Write $N = (n\mathrm{I} + P\cdot\boldsymbol{\sigma})/2$ and $\bar{N} = (m\mathrm{I} + Q\cdot\boldsymbol{\sigma})/2$. The state has

$$P = (u, 0, z), \qquad Q = (Ru, Ju, w), \qquad u = \sqrt{U} > 0, \qquad \Omega = \frac{1}{10}, \quad x = \frac{2191882124554491}{7927203751280000}. \tag{5}$$

For an exact and compact specification, define

$$g = (1 + r)/2, \qquad b = (c + r)/2, \qquad d = 33/10, \qquad f = 11/2, \tag{6}$$

$$L = \frac{4\Omega^2 + r^2 - c}{2(d/g - cf/b)}, \qquad x = L/\Omega, \tag{7}$$

$$z = \frac{g(b^2 + \Omega^2)}{L(g - b)}, \qquad w = \frac{b(g^2 + \Omega^2)}{L(g - b)}, \tag{8}$$

$$R = \frac{gb + \Omega^2}{b^2 + \Omega^2}, \qquad J = \frac{\Omega(b - g)}{b^2 + \Omega^2}, \tag{9}$$

$$U = \frac{r}{g}(d - z)\frac{b^2 + \Omega^2}{L(g - b)}, \tag{10}$$

$$n = \frac{175}{10} + \frac{1 - r}{2g}(d - z), \qquad m = \frac{197}{10} + \frac{c - r}{2b}(f - w). \tag{11}$$

These formulas uniquely determine all entries using rational arithmetic except for the positive square root u. In particular,

$$U = \frac{3683655029312150384608627360990063}{91493988989798101165668927156378564}, \quad R = \frac{1870833769}{690271369}, \quad J = -\frac{842352000}{690271369}. \tag{12}$$

Substitution in the eight Bloch equations gives zero residual. Exact positivity tests are

$$n > 0, \quad n^2 - z^2 - U > 0, \qquad m > 0, \quad m^2 - w^2 - U(R^2 + J^2) > 0. \tag{13}$$

They establish strict positive definiteness of both density matrices. The positive U establishes coherence. Appendix A gives the substitution explicitly, and the supplement verifies it without floating-point arithmetic.

3 Complete linearized operator and main result

A Fourier perturbation has dependence $e^{\mathrm{i}q\zeta}$. In physical Bloch coordinates $X = (a, p, \bar{a}, t) \in \mathbb{C}^8$, put $h = x(P - Q) - \Omega e_z$, $d_n = (1 - r)/2$, $d_b = (c - r)/2$, and define

$$\begin{aligned} A_{\mathrm{ph}}X = (&-ga - d_n p_z,\ h \times p - gp - d_n a e_z,\\ &- b\bar{a} - d_b t_z,\ h \times t - bt - d_b \bar{a} e_z), \end{aligned} \tag{14}$$

$$B_{\mathrm{ph}}y = (0, xy \times P, 0, xy \times Q), \qquad C_{\mathrm{ph}}X = p - t. \tag{15}$$

The Hilbert space is $\mathcal{H} = L^2([-1, 1], dv/2; \mathbb{C}^8)$ with the Euclidean Bloch norm. The full generator is the bounded operator

$$\mathcal{L}_q X(v) = (A_{\mathrm{ph}} - \mathrm{i}qv\mathrm{I})X(v) + B_{\mathrm{ph}}\big(\langle C_{\mathrm{ph}}X\rangle - v\,\langle vC_{\mathrm{ph}}X\rangle\big). \tag{16}$$

No angular truncation or restriction to off-diagonal entries is made in this definition.

Theorem 1. *At the exact state (5)–(7) of (1), the following statements hold.*

1. *For every real $|q| \le 2^{-62}$,*

$$\sup_{\tau \ge 0} \left\| e^{\tau \mathcal{L}_q} \right\|_{\mathcal{H} \to \mathcal{H}} \le 8370547. \tag{17}$$

The simple phase eigenvalue is real and even in q, with

$$\sigma_\phi(q) = -D_\phi q^2 + O(q^4), \qquad D_\phi > 0. \tag{18}$$

$$D_\phi = \frac{864103597051860981227428059103394991947954533150}{208678168337979529924684410968722713025006087091} > 0. \tag{19}$$

On this interval, $\sigma_\phi(q) \le -D_\phi q^2/2$; the remaining evolution decays with exponent at least $1/500000$.

2. *At $q = 3/64$, the same operator at the same state has a real eigenvalue satisfying*

$$\frac{16548364370979}{144115188075855872} < \sigma < \frac{51741326394403}{144115188075855872}. \tag{20}$$

Its eigenfunction is bounded in v and belongs to $\mathcal{H}$.

The first assertion includes $q = 0$ and arbitrary axial angular perturbations. A neutral flavor rotation remains at $q = 0$; boundedness is the asserted form of linear stability. For nonzero small q this mode is damped. The second assertion establishes linear exponential instability, not its nonlinear outcome. Real Hermitian perturbations are obtained by combining the conjugate q and $-q$ Fourier amplitudes.

4 Proof of long-wavelength boundedness

4.1 Rational coordinates and all angular sectors

Use $X_{\mathrm{ph}} = SX$, where

$$S = \mathrm{diag}(1, u, u, 1, 1, u, u, 1), \qquad T_m = \mathrm{diag}(u, u, 1), \tag{21}$$

and set $A = S^{-1}A_{\mathrm{ph}}S$, $B = S^{-1}B_{\mathrm{ph}}T_m$, $C = T_m^{-1}C_{\mathrm{ph}}S$. All three matrices are rational. Streaming commutes with S. Since $1/25 < U < 1$, $\|S\| \left\|S^{-1}\right\| < 5$. In the following proof all operator norms are in these transformed coordinates until the final multiplication by 5.

The exact moments $\langle 1\rangle = 1$, $\langle v\rangle = 0$, $\langle v^2\rangle = 1/3$ show that $L_0 = S^{-1}\mathcal{L}_0 S$ is an orthogonal angular direct sum of

$$M_0 = A + BC, \qquad M_1 = A - BC/3, \qquad A \quad (\ell \geq 2), \tag{22}$$

on the constant, dipole and higher Legendre sectors. This decomposition includes the infinite tail. The physical local symmetric part has eigenvalues $-1, -r, -g, -g, -c, -r, -b, -b$. Consequently

$$\left\|e^{\tau A}\right\| \leq 5e^{-\rho\tau}, \qquad \rho = r = 41/10900. \tag{23}$$

The precession is skew-adjoint in physical coordinates. It follows also that the local multiplication resolvent has no poles in the right half-plane for real q.

The phase vector in rational coordinates is

$$r_0 = (0, 0, 1, 0, 0, -J, R, 0)^T. \tag{24}$$

Solve $\ell_0 M_0 = 0$, $\ell_0 r_0 = 1$ and put $P = r_0\ell_0$, $Q = \mathrm{I} - P$. Exact elimination verifies these identities. For transparency, the finite-dimensional strict-stability certificate consists of symmetric rational matrices H_0, H_1 for $G_0 = M_0 - P$ and $G_1 = M_1$. Each has strictly positive exact LDL^T pivots, and

$$W_i = -(G_i^T H_i + H_i G_i), \quad w_i = \min_j \left((W_i)_{jj} - \sum_{k\neq j} |(W_i)_{jk}| \right) > 99/100. \tag{25}$$

The rational entries are supplied in `data/lyapunov-certificate.json`. They are finite witnesses to inequalities, not fitted parameters. Define

$$h_i^+ = \max(\|H_i\|_1, \|H_i\|_\infty), \qquad h_i^- = 1/\max(\left\|H_i^{-1}\right\|_1, \left\|H_i^{-1}\right\|_\infty). \tag{26}$$

Exact checks give $\sqrt{h_0^+/h_0^-} \leq 605$, $\sqrt{h_1^+/h_1^-} \leq 18$, and $w_i/(2h_i^+) \geq \gamma = 1/250000$. Differentiating $y^* H_i y$ proves

$$\left\|e^{\tau G_i}\right\| \leq \sqrt{h_i^+/h_i^-}\, e^{-w_i\tau/(2h_i^+)}. \tag{27}$$

Thus M_0 has a simple phase zero and a strictly decaying complement; this also follows from its characteristic polynomial and exact Routh signs. The projection bounds are $\|P\| \leq p = 53$ and $\|Q\| \leq b_0 = 52$. Let P_0 equal P on constant angular functions and zero otherwise, with $Q_0 = \mathrm{I} - P_0$. Equations (22)–(25) yield

$$\left\|e^{\tau L_0} Q_0\right\| \leq M e^{-\gamma\tau}, \qquad M = \max(605 \cdot 52, 18, 5) = 31460. \tag{28}$$

This is a semigroup estimate, not an inference from a sampled spectrum.

4.2 Phase continuation and diffusion

Write $L_q = L_0 + qV$, $V = -\mathrm{i}v\mathrm{I}$, $\|V\| = 1$. On $|z| = h_0 = \gamma/2 = 1/500000$, the phase/complement resolvent decomposition and (28) give

$$\left\|(z - L_0)^{-1}\right\| \le p/h_0 + M/(\gamma - h_0) = R_0 = 15756500000. \tag{29}$$

Set $R_q = (4R_0)^{-1} = 1/63026000000$. For complex $|q| \le R_q$, a Neumann series keeps this circle in the resolvent set. Its Riesz projection $P(q)$ has rank one, and the enclosed eigenvalue $\sigma_\phi(q)$ is analytic, with $|\sigma_\phi(q)| < h_0$. Analyticity extends beyond the closed chosen disk because the Neumann product there is only $1/4$.

Angular reflection maps L_q to L_{-q}; complex conjugation maps L_q to $L_{-\bar{q}}$. Uniqueness of the enclosed eigenvalue makes it even and real for real q. Cauchy estimates therefore give, for $|q| \le R_q/2$,

$$|\sigma_\phi(q) - a_2 q^2| \le C_4 |q|^4, \qquad C_4 = \frac{4h_0}{3R_q^4}. \tag{30}$$

To find a_2, normalize the leading constant eigenvector to r_0. The first-order equation is $L_0 X_1 = \mathrm{i}v r_0$, hence $X_1 = \mathrm{i}v M_1^{-1} r_0$. Projecting the second-order equation with ℓ_0 and averaging v^2 gives

$$a_2 = \tfrac{1}{3}\ell_0 M_1^{-1} r_0 = -D_\phi. \tag{31}$$

Exact substitution gives (19); its decimal value is approximately 4.140843308785137. The supplement independently obtains the same coefficient from the six-moment determinant's power series. Higher angular sectors are not dropped: their effects are covered by the full-operator remainder (30).

4.3 The moving complement

The resolvent identity bounds

$$\|P(q) - P_0\| \le \delta(q) = \frac{h_0 R_0^2 |q|}{1 - R_0|q|}. \tag{32}$$

Define $Q(q) = \mathrm{I} - P(q)$ and

$$T(q) = P(q)P_0 + Q(q)Q_0 = \mathrm{I} + [P(q) - P_0](2P_0 - \mathrm{I}). \tag{33}$$

If $\eta = (2p+1)\delta < 1$, then $\|T\| \le 1 + \eta$ and $\left\|T^{-1}\right\| \le (1-\eta)^{-1}$. This invertible map carries the fixed splitting to the moving phase and complement spaces. For $y \in \operatorname{ran} Q_0$, $Ty = Q(q)y$, so commutation of $Q(q)$ with L_q gives the exact restricted identity

$$T^{-1} L_q T y = L_0 y + q T^{-1} Q(q) V y. \tag{34}$$

The perturbation has norm at most $E = |q|(b_0 + \delta)/(1 - \eta)$. Duhamel's formula and Gronwall's inequality applied to (28) bound the transformed complementary semigroup by $Me^{-(\gamma - ME)\tau}$. Returning to the whole space yields

$$\left\|e^{\tau L_q} Q(q)\right\| \le \frac{1+\eta}{1-\eta} M(b_0 + \delta) e^{-(\gamma - ME)\tau}. \tag{35}$$

At $q_* = 2^{-62}$ the following inequalities hold exactly:

$$q_* \le R_q/2, \quad C_4 q_*^2 \le D_\phi/2, \quad \eta(q_*) \le 1/2, \quad ME(q_*) \le \gamma/2. \tag{36}$$

Their bounding functions increase with $|q|$, so they hold throughout the claimed interval. The phase contribution is bounded by $p + \delta$. The physical bound is consequently

$$5\left[p + \delta(q_*) + \frac{1+\eta(q_*)}{1-\eta(q_*)} M(b_0 + \delta(q_*))\right] \le 8370547. \tag{37}$$

This proves the first part of Theorem 1, including neutrality at $q = 0$. Keeping the moving phase projection is essential; simply deleting the phase direction at $q \neq 0$ would not prove this statement.

5 Proof of a finite-wavelength continuum eigenvalue

5.1 Finite-rank reduction without angular truncation

Fix $q = 3/64$. In rational coordinates define

$$R(\sigma, v) = [(\sigma + \mathrm{i}qv)\mathrm{I} - A]^{-1}, \quad J_j(\sigma) = \left\langle v^j C R(\sigma, v) B \right\rangle \quad (j = 0, 1, 2), \tag{38}$$

$$H(\sigma) = \begin{pmatrix} \mathrm{I}_3 - J_0 & J_1 \\ -J_1 & \mathrm{I}_3 + J_2 \end{pmatrix}, \qquad F(\sigma) = \det H(\sigma). \tag{39}$$

For real q and $\operatorname{Re}\sigma \ge 0$, physical local dissipativity implies

$$\left\| [(\sigma + \mathrm{i}qv)\mathrm{I} - A_{\mathrm{ph}}]^{-1} \right\|_2 \le (\operatorname{Re}\sigma + \rho)^{-1}. \tag{40}$$

Thus the determinant is analytic throughout the right half-plane and has no hidden multiplication-spectrum poles there.

If $H(\sigma)(m_0, m_1)^T = 0$, then

$$X(v) = R(\sigma, v) B(m_0 - v m_1) \tag{41}$$

satisfies $\langle CX \rangle = m_0$ and $\langle vCX \rangle = m_1$, exactly by the two block rows of (39). Hence it is a continuum eigenfunction. Conversely, an eigenfunction in this half-plane gives these moments; zero moments would force $X = 0$. This is an exact finite-rank reduction of the angular integral, not discretization of the angular space.

5.2 Validated integral and root count

Consider the rectangle centered at

$$\sigma_0 = \frac{34144845382691}{144115188075855872}, \qquad |\operatorname{Re}(\sigma - \sigma_0)| \le h, \quad |\operatorname{Im}\sigma| \le h, \qquad h = \frac{4097}{33554432}. \tag{42}$$

It lies strictly in the right half-plane. Its center was motivated by numerical exploration, but only the following enclosures enter the proof.

Divide $[-1, 1]$ into 128 panels of half-width $a = 1/128$, centered at v_c. Compute a directed interval inverse $R_c = [(\sigma_0 + \mathrm{i}qv_c)\mathrm{I} - A]^{-1}$. With $\delta = \sigma - \sigma_0$ and $t = v - v_c$,

$$R(\sigma, v) = \sum_{n=0}^{N} (-1)^n (\delta + \mathrm{i}qt)^n R_c^{n+1} + \mathcal{E}_N. \tag{43}$$

If r_c bounds the induced infinity norm of R_c, then

$$\theta_c = r_c(2h + |q|a) < 1, \qquad \|\mathcal{E}_N\|_\infty \le \frac{r_c \theta_c^{N+1}}{1 - \theta_c}. \tag{44}$$

Expanding $(\delta + \mathrm{i}qt)^n$ and $(v_c + t)^j$ and integrating monomials exactly gives polynomial interval coefficients for $H_N(\delta)$. The normalized integral of t^e on a panel is zero for odd e, and $a^{e+1}/(e+1)$ for even e. Since $\|C\|_\infty = 2$ and $|v|^j \le 1$, each entry of $H - H_N$ is bounded by

$$e_N = \frac{2\,\|B\|_\infty}{128} \sum_c \frac{r_c \theta_c^{N+1}}{1-\theta_c}. \tag{45}$$

Both σ and v variation are covered by this estimate.

Directed 25-digit decimal arithmetic, including every coefficient-rounding operation, gives the bounds in Table 1. A second calculation at $N = 32$ reconstructs the operator directly from the 2×2 commutator and anticommutator derivatives; it uses the same interval arithmetic and integration formula, so it is not a separate mathematical proof technique. The public code regenerates both sets of coefficients from the exact state rather than trusting stored success flags.

Quantity	Order $N = 16$	Order $N = 32$
Angular panels	128	128
Maximum θ_c	< 0.129767	< 0.129767
Uniform entry tail e_N	$< 4.166444 \times 10^{-14}$	$< 1.966585 \times 10^{-28}$
Whole contour edges	4	4
Determinant winding	1	1

Table 1: Bounds for the continuous-angle integral on the entire rectangle (42). The tail bounds are in addition to the coefficient-rounding intervals.

On each entire edge, interval Gaussian elimination encloses $\det H$ after inflating every matrix entry by e_N. The convex rectangular hull of this enclosure and both endpoint enclosures excludes zero. On the bottom, right, top and left edges, respectively, the hull lies in the negative imaginary, positive real, positive imaginary and negative real half-planes. The determinant curve is therefore homotopic, without crossing zero, to the polygon of rational endpoint midpoints. Exact ray-crossing arithmetic gives winding number $+1$ for this polygon. Since (40) excludes poles, the argument principle gives exactly one zero, counting multiplicity, inside the rectangle.

Deleting row and column 0 of H gives a 5×5 minor that is nonzero on the whole rectangle. For $N = 16$, its real determinant enclosure lies in

$$[0.055424393499086033016906 79,\ 0.055682523592679089151373 38], \tag{46}$$

which excludes zero irrespective of its imaginary part. The $N = 32$ calculation independently re-establishes the minor condition. Set the first moment component to 1 and solve the other five rows using this invertible minor. At the zero of F, the remaining row vanishes by the Schur determinant identity, giving an exact nonzero null vector. Equation (41) then reconstructs a bounded eigenfunction. It cannot be zero because its first moment component is 1.

Finally, real A, B, C and the change $v \mapsto -v$ imply

$$H(\bar{\sigma}) = D_c \overline{H(\sigma)} D_c, \qquad D_c = \mathrm{diag}(\mathrm{I}_3, -\mathrm{I}_3). \tag{47}$$

The rectangle is invariant under conjugation and contains only one zero. That zero must be real. Subtracting and adding its half-width from its center gives (20). Reality is concluded from uniqueness and symmetry after the complex root count; it is not an input to the numerical enclosure. This proves the second part of Theorem 1.

6 Interpretation and limitations

The theorem rules out an implication within the stated model: even complete zero-wave-number angular stability, together with bounded evolution on a nonzero long-wavelength interval, does not imply stability at every spatial wave number. The counterexample cannot be explained by omitting the dipole sector at $q = 0$, omitting density perturbations, or mistaking a finite angular-grid eigenvalue for a continuum mode. The long-wavelength phase response is diffusive and damping; the proof does not identify the finite-wave-number root as a continuation of that phase mode.

The large $K = 8370547$ is an upper bound produced by coarse norm and intertwiner estimates, not a prediction of transient amplification. Similarly, 2^{-62} is a sufficient proof radius, not the actual stable-band edge. The theorem leaves almost all intervening wave numbers unclassified and gives no instability threshold or full unstable band. A numerical root estimate near 0.00023692746 may guide computation, but only (20) is a rigorous eigenvalue enclosure.

The result concerns a single equilibrium at exact rate and target ratios. It does not establish robustness to rounding of astrophysical inputs, identify a physical interaction strength or wavelength, or show that a source trajectory approaches this state. Finite vacuum mixing, multiple energies, nonaxial perturbations, realistic scattering and dynamical reservoir feedback are absent. Linear growth alone does not determine nonlinear saturation or the eventual flavor composition. In particular, this example neither invalidates a specific transport closure in its calibrated regime nor quantifies an astrophysical observable. Its significance is a controlled test case for the logical sufficiency of local stability diagnostics in a coherent repopulation model.

7 Conclusion

An explicitly specified coherent relative equilibrium can be stable in the complete homogeneous axial-continuum problem and throughout a rigorously controlled long-wavelength interval, yet unstable at a finite wave number. The proof keeps the phase mode, all density components and the angular continuum. This supplies a reproducible same-state counterexample in the fixed-reservoir model and identifies precisely which stronger physical conclusions require additional evidence.

Reproducibility

The accompanying `neutrino-stability-supplement.zip` contains the exact state and rational operator, Lyapunov and phase certificates, continuous-angle interval coefficients, and Python 3 standard-library scripts. Run `python3 verify_state.py`, `python3 verify_longwave.py`, and `python3 verify_continuum.py` without Python's `-O` option. The first reconstructs the equilibrium and verifies all density-matrix Jacobian entries. The second checks the finite rational inequalities in the long-wavelength argument. The third regenerates continuous-angle integration and contour/minor enclosures at orders 16 and 32. These computations check finite witnesses used in the analytical arguments; they do not automate the functional-analytic implications or establish physical applicability. No observational or simulated astrophysical data are used.

A Equilibrium residuals and rational operator

To check the construction without manipulating square roots, set

$$h'_x = x(1 - R), \quad h'_y = -xJ, \quad h_z = x(z - w) - \Omega. \tag{48}$$

Here $h_x = uh'_x$ and $h_y = uh'_y$. The eight residuals, dividing transverse rows by u, are

$$g(17.5 - n) + d_n(3.3 - z), \tag{49}$$
$$h'_y z - g, \qquad h_z - h'_x z, \tag{50}$$
$$-h'_y U - gz + 3.3g + d_n(17.5 - n), \tag{51}$$
$$b(19.7 - m) + d_b(5.5 - w), \tag{52}$$
$$h'_y w - h_z J - bR, \qquad h_z R - h'_x w - bJ, \tag{53}$$
$$(h'_x J - h'_y R)U - bw + 5.5b + d_b(19.7 - m). \tag{54}$$

Every decimal here is terminating and exact. Substitution of (7) gives zero. The rational local matrix has two diagonal 4×4 blocks

$$A(G, \Delta) = \begin{pmatrix} -G & 0 & 0 & -\Delta \\ 0 & -G & -h_z & h'_y \\ 0 & h_z & -G & -h'_x \\ -\Delta & -Uh'_y & Uh'_x & -G \end{pmatrix}, \tag{55}$$

with $(G, \Delta) = (g, d_n)$ and (b, d_b). Furthermore,

$$B = x \begin{pmatrix} 0 & 0 & 0 \\ 0 & z & 0 \\ -z & 0 & 1 \\ 0 & -U & 0 \\ 0 & 0 & 0 \\ 0 & w & -J \\ -w & 0 & R \\ UJ & -UR & 0 \end{pmatrix}, \qquad CX = (p_1 - t_1, p_2 - t_2, p_3 - t_3)^T, \tag{56}$$

where the transverse entries of X are in rational coordinates. These formulas independently specify the entire operator used by the certificates.

B Certificate logic and checks

The exact state, matrices and finite witnesses are enough to repeat every numerical assertion of the proof. For the long-wavelength part, positive LDL^T pivots establish $H_i > 0$; Gershgorin bounds establish (25); induced matrix norms bound the Euclidean norm because $\|M\|_2 \le \sqrt{\|M\|_1 \, \|M\|_\infty}$. Exact inverses give the lower metric bounds. The code checks the phase vectors, projections, coefficient (19), similarity bounds, remainder and complement inequalities, and final ceiling in (17). Stored equilibrium residuals are independently recalculated by `verify_state.py`.

For the finite-wave-number part, decimal intervals use directed floor and ceiling arithmetic for every elementary operation. Complex intervals are rectangles; division is permitted only when the denominator's squared modulus has a strictly positive lower endpoint. Interval row elimination encloses the inverse or determinant of each exact matrix represented by the input. Exact identities for eliminated entries are used only where elimination makes them identically zero. A Neumann tail bounds the terms not integrated. Determinant hulls cover whole edges, so the root count does not rely on sampled phase values. Saved coefficient and hull files permit inspection, but `verify_continuum.py` does not need to accept their claimed winding or positivity: it recomputes those enclosures.

The factor $1/128$ in (45) is the normalized measure of each panel. The contour counts a zero only after excluding local resolvent poles. The minor supplies a nonzero moment vector, and moment reconstruction supplies a nonzero L^2 eigenfunction. These separate steps distinguish the certificate from a determinant minimum or an apparent eigenvalue in a Legendre truncation.

References

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