Kerr Electromagnetic Error Bounds and a Limitation of Cubic Reference Fields

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Abstract

A computed electromagnetic pulse near a rotating black hole is useful only if its error can be related to a physical observable. We construct that relation for Maxwell fields on a fixed subextremal Kerr exterior and on a finite core with a power-normalized outer boundary. The construction retains electric and magnetic charge sectors, conserved interior currents, weak horizon loss, and incoming histories that are only square integrable in time. For spin $a/M = 0.9$, a verified coefficient calculation bounds the signed production of a fixed positive energy by a constant between $0.499180315738376/M$ and $0.525452399065317/M$. A causal-region argument restricts its effective duration to $80M$ for a specified forward-supported, zero-initial-error class, while keeping the actual boundary output and every cap loss. The resulting sufficient residual target is nevertheless severe: approximately 3.20×10^{-22} for a 10^{-3} output-amplitude tolerance in the normalized example. A nonzero cubic reference field has a complete but uninformative error enclosure. More decisively, an exact fourth-degree Legendre moment proves that neither this family nor its prescribed single midpoint refinement can attain that sufficient residual target, whatever their coefficients. This obstruction concerns two approximation families and one certificate; it neither bounds the true output error from below nor excludes a physical response or other numerical methods.

1 Introduction

A finite calculation of a wave on a black-hole background has two different tasks. It must produce an approximate field, and it must bound the difference between that field and the physical solution in an observable norm. A small residual at fitting points does not accomplish the second task. Nor does an energy identity alone establish that rough incoming data produce a solution with a usable boundary trace. These distinctions become consequential when a conservative error estimate amplifies a residual by a large exponential factor.

This paper develops a connected construction for forced Maxwell evolution and a finite-radius power observation on fixed Kerr spacetime. A variational charge lift first converts a specified conserved electric current into a compatible forcing of the closed-field equations. A horizon-penetrating construction then supplies the actual forward evolution and its generator domain. At a finite outer boundary, a separate weak boundary construction produces the incoming-to-outgoing power map. It prescribes the two incoming radiative components and retains the two normal charge directions without treating them as incoming radiation.

The quantitative part fixes $a/M = 0.9$ and an outer radius $1000M$. We enclose the signed energy-production coefficient for one specified positive multiplier. We then prove a localized estimate in which the deformation term is active for at most $80M$ within a containing causal region. This is a statement about the energy balance in that region, not a bound on how long radiation can remain trapped. The observable remains the physical outer power, including all output before the selected final time.

Finally, we construct a closed-potential reference for one normalized, compact pulse and evaluate its full physical residual. Its error enclosure is too wide to establish a useful return. An exact dual moment gives a stronger, sharply limited conclusion: all coefficient choices in two specified cubic families fail the sufficient residual-norm target furnished by the energy estimate. This identifies a limitation of that approximation-and-certificate combination without converting a pessimistic upper bound into a physical impossibility result.

The underlying analytic ingredients—a Riesz charge lift, symmetric-hyperbolic regularization, dissipative boundary realization, and energy comparison—are supporting constructions. No literature-wide priority claim is made. Homogeneous Kerr boundedness results, including the distinct weighted component norms in Theorem 5.1 of [1], provide relevant context; they are not used here as an unproved forcing or finite-port theorem. All estimates needed below are derived in the stated physical energy and boundary spaces.

2 Geometry, fields, and the observable

We use rationalized field units with $c = G = 1$, spacetime signature $(-+++)$, and stationary ingoing Cartesian Kerr–Schild coordinates (t, x, y, z). Let $M > 0$, $|a| < M$, and

$$r_\pm = M \pm \sqrt{M^2 - a^2}, \qquad r^4 - (x^2 + y^2 + z^2 - a^2)r^2 - a^2 z^2 = 0, \tag{1}$$

$$H_K = \frac{Mr^3}{r^4 + a^2 z^2}, \qquad \ell = \left(\frac{rx + ay}{r^2 + a^2}, \frac{ry - ax}{r^2 + a^2}, \frac{z}{r} \right), \qquad |\ell|_{\mathbb{R}^3} = 1, \tag{2}$$

$$g = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + 2H_K(\mathrm{d}t + \ell_i \mathrm{d}x^i)^2. \tag{3}$$

The slice metric, lapse, and shift are

$$h = I + 2H_K \ell \ell^T, \qquad \alpha = (1 + 2H_K)^{-1/2}, \qquad \beta = \frac{2H_K}{1 + 2H_K} \ell. \tag{4}$$

The spatial exterior is $\Sigma = \{r > r_+\}$; its horizon is a characteristic outflow boundary. The core is $\Omega = \{r_+ < r < R\}$ with $R = 1000M$. Spatial orientation is the asymptotic Cartesian orientation. The spatial Hodge star $*$, all covariant norms, and $\mathrm{d}V_h$ refer to h.

The state $F = (D, B)$ consists of electric and magnetic flux two-forms. Write $u = *D$, $v = *B$ and identify one-forms with vectors using h when taking cross products. The unconstrained Hilbert norm is

$$\|F\|^2_{\mathcal{H}_0} = \int (|u|^2_h + |v|^2_h) \mathrm{d}V_h, \qquad \mathcal{K} = \{(D, B) \in \mathcal{H}_0 : \mathrm{d}D = \mathrm{d}B = 0\}, \tag{5}$$

where the equations are distributional in the open spatial domain. In particular $\mathcal{K}$ includes fields with nonzero electric or magnetic flux through the horizon. For an electric charge $q = \rho \mathrm{d}V_h$ and coordinate current two-form I (the flux of $\alpha j - \rho\beta$), physical Maxwell equations are

$$D_t = \mathrm{d}(\alpha * B + \iota_\beta D) - I, \qquad \mathrm{d}D = q, \tag{6}$$

$$B_t = -\mathrm{d}(\alpha * D - \iota_\beta B), \qquad \mathrm{d}B = 0, \qquad q_t + \mathrm{d}I = 0. \tag{7}$$

Let $\mathcal{A}$ denote the source-free differential expression in these equations. Its maximal differential domain comprises the L^2 pairs for which $\mathcal{A}F \in L^2$ distributionally. A physical generator is constructed below; it is not defined by simply identifying it with that maximal expression.

2.1 A positive energy with the physical stationary power at the port

Define $p(s) = e^{-1/s}$ for $s > 0$ and $p(s) = 0$ otherwise, and put

$$S(s) = \frac{p(s)}{p(s) + p(1 - s)}, \qquad \chi(r) = 1 - S\left(\frac{r - 4M}{4M} \right), \tag{8}$$
$$w = -\chi\beta, \qquad b = (1 - \chi)\beta, \qquad X = \partial_t + w = \alpha n + b.$$

Here n is the future unit slice normal. This fixed smooth X equals αn near the horizon and the stationary Killing field $T = \partial_t$ for $r \geq 8M$. For $r \geq 4M$, $|\beta|_h/\alpha = 2H_K \leq 1/2$, so X is timelike throughout the core.

Set

$$E_T = \alpha u + \beta \times v, \qquad H_T = \alpha v - \beta \times u, \tag{9}$$

$$E_X = \alpha u + b \times v, \qquad H_X = \alpha v - b \times u, \qquad C\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} E_X \\ H_X \end{pmatrix}, \tag{10}$$

$$e_X = \tfrac{1}{2}(u \cdot E_X + v \cdot H_X) = \alpha e - b \cdot (u \times v), \quad e = \tfrac{1}{2}(|u|^2 + |v|^2), \tag{11}$$

$$E_X(F) = \int e_X \mathrm{d}V_h = \tfrac{1}{2}\|F\|^2_{\mathcal{H}}. \tag{12}$$

The eigenvalues of C are $\alpha, \alpha \pm |b|$, with multiplicities. In particular $e/3 \leq e_X \leq 2e$, so $\mathcal{H}$ and $\mathcal{H}_0$ have equivalent norms.

At the port, let $q_T = \sqrt{-g(T,T)}$ and, for tangent vectors e_A to a port cut, define its observer screen metric

$$\gamma_{AB} = g(e_A, e_B) + q_T^{-2} g(T, e_A) g(T, e_B). \tag{13}$$

Let J_γ be the positive 90° rotation of screen covectors. Tangential restrictions of E_T, H_T determine incoming and outgoing covectors

$$f = \tfrac{1}{2}(E_T + J_\gamma H_T), \qquad y = \tfrac{1}{2}(E_T - J_\gamma H_T). \tag{14}$$

The physical outward stationary-energy power is exactly

$$E_T \wedge H_T = (|y|^2_\gamma - |f|^2_\gamma)\mathrm{d}A_\gamma. \tag{15}$$

Indeed the worldtube measure is $q_T \mathrm{d}t\mathrm{d}A_\gamma$, and E_T, H_T are q_T times the stationary observer's screen fields. These factors give precisely the power in (15), with time t. Denote the resulting space of histories by $\mathcal{P} = L^2(\mathrm{d}t\mathrm{d}A_\gamma; T^*S^2)$. The boundary quadratic form has two incoming, two outgoing, and two zero directions. The zero directions are normal electric and magnetic components; they are not additional prescribed radiative data. We assert no arbitrary full-field L^2 trace on the timelike port.

Real fields may be used throughout. For the numerical example we complexify the linear equations, use Hermitian norms and real parts of work pairings, and normalize the complex power. This is equivalent to summing the powers of two real quadrature fields; no additional factor of two is inserted later.

3 Conserved currents and physical forward evolution

The source construction in this section concerns the full exterior. Fix a compact annulus $K_s \Subset \Sigma$. Assume $\rho \in L^1(0,T;L^2_h)$ and $I \in L^1(0,T;L^2_h(\Lambda^2))$ have support in K_s. Let V be the completion of $C^\infty_c(\Sigma)$ in $\|\mathrm{d}\phi\|_{L^2_h}$ and V^* its Hilbert dual. Require

$$q \in W^{1,1}(0,T;V^*), \qquad q_t + \mathrm{d}I = 0. \tag{16}$$

The zero horizon trace in this auxiliary Dirichlet space is a choice for a decomposition, not a condition on the physical field.

Proposition 3.1 (Compatible current lift)**.** *There is a linear map $L : V^* \to L^2_h(\Lambda^2)$ with $\mathrm{d}Lq = q$ and $\|Lq\|_2 = \|q\|_{V^*}$. The map $P = 1 - L\mathrm{d}$ on two-forms is the orthogonal projection onto closed fields. For the source class (16),*

$$\mathcal{G}(q,I) = \mathcal{A}(Lq, 0) - (PI, 0) \in L^1(0,T;\mathcal{K}), \quad \|\mathcal{G}\|_{\mathcal{H}_0} \leq \|I\|_2 + C_\rho\|\rho\|_2 + C_q\|q\|_{V^*}, \tag{17}$$

with explicit finite geometry coefficients specified in Appendix A. The lift belongs to the maximal differential domain. Physical Maxwell evolution is equivalent to

$$z_t = \mathcal{A}z + \mathcal{G}(q, I), \qquad F = z + (Lq, 0). \tag{18}$$

Proof. Riesz representation gives a unique $\phi_q \in V$ satisfying $\int \langle \mathrm{d}\phi_q, \mathrm{d}\psi\rangle \mathrm{d}V_h = \langle q, \psi\rangle$ for $\psi \in V$. Set $Lq = - * \mathrm{d}\phi_q$. Integration by parts proves $\mathrm{d}Lq = q$ and the norm equality. Since $\langle \mathrm{d}I, \psi\rangle = -\int \langle *I, \mathrm{d}\psi\rangle \mathrm{d}V_h$, the field $L\mathrm{d}I$ is the orthogonal gradient component of I, proving the assertions about P. Bounded linearity gives $(Lq)_t = -L\mathrm{d}I$ in $L^1_t L^2$. The Hessian estimate in Appendix A implies (17); compatibility follows from $\mathrm{d}^2 = 0$. Substitution into (7) proves the equivalence. Conversely applying d to those equations propagates $\mathrm{d}D - q = 0$ and $\mathrm{d}B = 0$ from the initial constraints. □

Theorem 3.2 (All-charge exterior evolution)**.** *For each fixed subextremal Kerr metric there is a strongly continuous forward semigroup $S(t)$ on the full space $\mathcal{K}$ of (5). Every $z_0 \in \mathcal{K}$ and $g \in L^1(0, T; \mathcal{K})$ has a unique $z \in C([0, T]; \mathcal{K})$ solving $z_t = \mathcal{A}z + g$ distributionally, and*

$$z(t) = S(t)z_0 + \int_0^t S(t - s)g(s)\mathrm{d}s, \tag{19}$$

$$\|z(t)\|_{\mathcal{H}_0} \le e^{\omega t}\|z_0\|_{\mathcal{H}_0} + \int_0^t e^{\omega(t-s)}\|g(s)\|_{\mathcal{H}_0}\mathrm{d}s, \qquad \omega = 482/M \text{ is admissible.} \tag{20}$$

Its actual generator is the strong forward derivative of $S(t)$. It contains $H^1 \cap \mathcal{K}$ and is contained in the maximal differential domain intersected with $\mathcal{K}$; equality with that maximal domain is not asserted.

Proof. Off the constraint space use the augmented operator

$$\widetilde{\mathcal{A}}(D, B) = (\mathrm{d}(\alpha * B) + \mathcal{L}_\beta D, -\mathrm{d}(\alpha * D) + \mathcal{L}_\beta B). \tag{21}$$

Cartan's identity gives $\widetilde{\mathcal{A}} - \mathcal{A} = (\iota_\beta \mathrm{d}D, \iota_\beta \mathrm{d}B)$. In an h-orthonormal frame its first-order symbol has diagonal part $(\beta \cdot \xi)I$ and off-diagonal parts $\alpha\,\mathrm{Cross}(\xi)$ and $-\alpha\,\mathrm{Cross}(\xi)$, hence is symmetric. The local energy identity is

$$e_t = \mathrm{div}_h J + b_0(F) + \langle F, g\rangle, \quad J = \beta e - \alpha u \times v, \quad b_0(F) \le (|\operatorname{div}\beta| + 4\|\nabla\beta\|_{\mathrm{op}} + |\mathrm{d}\alpha|)e. \tag{22}$$

For completeness the finite coefficient follows directly from (3). The inequalities $|\nabla_{\mathbb{R}^3} r| \le \sqrt{2}$ and $|a| \le r$ give $|\nabla_{\mathbb{R}^3} H_K| < 12M/r^2$ and $\|\partial\ell\|_{\mathrm{op}} < 5/r$. Hence $\|\partial h\|_{\mathrm{op}} < 44M/r^2$, and the Cartesian connection bilinear map has norm at most $66M/r^2$. Writing $f_K = 2H_K/(1+2H_K)$, use $f_K \le 2/3$, $f_K \le 2H_K$ and $|f_K'| \le 2$ to obtain $\|\partial\beta\|_{\mathrm{op}} \le 34M/r^2$. The connection contribution is at most $44M/r^2$, giving $\|\nabla\beta\|_{\mathrm{op,h}} \le 78\sqrt{3}M/r^2 < 136M/r^2$. Also $|\mathrm{d}\alpha|_h < 12M/r^2$ and $|\operatorname{div}\beta| \le 3\|\nabla\beta\|_{\mathrm{op,h}}$. Equation (22) therefore has coefficient at most $964M/r^2 \le 964/M$, proving (20).

Smooth the metric, lapse, and shift strictly inside the horizon, keeping them positive and unchanged on $r \ge 3M/4$. Conjugating the stationary positive density to Euclidean L^2 gives $U_t = (S^j\partial_j + B)U + f$ with symmetric S^j and bounded smooth coefficients. With an even convolution J_ε, the bounded-operator equations $U_\varepsilon' = J_\varepsilon(S^j\partial_j + B)J_\varepsilon U_\varepsilon + J_\varepsilon f$ have uniform energy estimates, also after up to four spatial derivatives for smooth H^4 data. Weak compactness, local compact embedding and the H^3 time-derivative bound give a distributional solution. The difference energy estimate extends it to arbitrary L^2 data and $L^1_t L^2$ sources by completion in $C_t L^2$. A first-order mollifier commutator is bounded by the Lipschitz coefficient norm and converges to zero first on smooth fields, then by density on L^2 fields. This proves the energy inequality and uniqueness for the completed weak class.

Restrict this solution to the exterior. The identities

$$\ell \cdot \mathrm{d}r = 1, \quad \beta^r = \frac{2H_K}{1+2H_K}, \quad |\mathrm{d}r|_h^2 - \frac{(\beta^r)^2}{\alpha^2} = \frac{(r-r_+)(r-r_-)}{r^2 + a^2\cos^2\theta} \tag{23}$$

imply $J \cdot \mathrm{d}r \geq -18(r-r_+)e/M$ in a fixed horizon collar. A radial cutoff varying over width ε therefore has an adverse contribution at most a fixed multiple of the spacetime energy in that collar, which tends to zero. Outer cutoffs have error bounded by $O(R^{-1})$ times shell energy. Thus exterior uniqueness and (20) hold without a horizon field trace; the restriction is independent of the artificial interior extension.

For the constraints $c = \mathrm{d}F$, the extended equation gives $c_t = \mathcal{L}_\beta c + \mathrm{d}g$. If Φ_t is the flow of the extended shift, its solution in H^{-1} is $c(t) = \Phi_t^* c(0) + \int_0^t \Phi_{t-s}^* \mathrm{d}g(s)\mathrm{d}s$. Pullback transports support by Φ_{-t}. Since $-\beta$ points into the hole at the horizon, constraints introduced by zero extension remain inside it. The exterior field is therefore closed for closed data and sources, with arbitrary charges retained.

Stationarity and uniqueness give the semigroup law. For any $z \in \mathcal{K}$, its time average $z_\varepsilon = \varepsilon^{-1}\int_0^\varepsilon S(s)z\mathrm{d}s$ lies in the generator domain and has generator $(S(\varepsilon)z - z)/\varepsilon$. These averages prove density. Passing to limits in $S(t)z - z = \int_0^t S(s)Az\mathrm{d}s$ proves closedness. Distributional testing identifies $Az = \mathcal{A}z$. A smooth collar reflection extends exterior H^1 fields to H^1 on the full space, proving the stated domain inclusion. Finally the same endpoint calculation for $\int_0^t S(s)g\mathrm{d}s$ proves (19) for step sources; L^1 approximation and (20) prove it for the declared source class. □

3.1 Charges and the source-to-port qualification

For a smooth cutoff χ_H equal to one near the horizon and zero far away, define

$$Q_H(D, q) = \langle q, \chi_H \rangle + \int \langle *D, \mathrm{d}\chi_H \rangle \mathrm{d}V_h. \tag{24}$$

For compact sources the first pairing uses any interior cutoff equal to one on their support. Gauss law makes (24) independent of the cutoff; it agrees with flux in the outward-from-exterior orientation when traces exist. The infinity flux satisfies $Q_\infty + Q_H = \langle q, 1 \rangle$. Magnetic charge uses $Q_H(B, 0)$. For closed forcing,

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_H(z_E, 0) = \int \langle *g_E, \mathrm{d}\chi_H \rangle \mathrm{d}V_h, \tag{25}$$

and likewise magnetically. Thus arbitrary compatible forcing can change charge. For (17), however,

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_H(Lq, q) = \langle *PI, \mathrm{d}\chi_H \rangle_{L^2}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} Q_H(z_E, 0) = -\langle *PI, \mathrm{d}\chi_H \rangle_{L^2}. \tag{26}$$

The physical electric charge is constant, as are the magnetic and infinity charges for this compact interior-current class. The compensating lift charge must not be omitted.

The source theorem alone does not give L^2 time-port traces of Lq. To use it with a physical port input one must establish $f_z = f_{\mathrm{phys}} - f_{Lq}$ and $y_{\mathrm{phys}} = y_z + y_{Lq}$ in the port spaces. A sufficient additional assumption is $\rho \in L_t^2 L_h^2$, $q \in L_t^\infty V^*$ and the lift's H^1 regularity in a fixed port collar. The Hessian estimate and

$$\|v|_{r=R}\|_2^2 \leq C_{\mathrm{col}}(\ell^{-1}\|v\|_{2,\mathrm{col}}^2 + \ell\|\nabla v\|_{2,\mathrm{col}}^2) \tag{27}$$

then supply those traces. Here C_{col} includes twice the collar Jacobian ratio and the bounded conversion to the screen norm. Equation (27) follows by averaging $|v(0)|^2 \leq 2|v(s)|^2 + 2s\int_0^s |\nabla_n v|^2$. The original $L_t^1 L^2$ density assumption alone does not imply this stronger boundary assertion.

4 A weak core boundary map

Use $\mathcal{H}$ and $\mathcal{K}$ on Ω. For construction off the constraints take

$$D_t = \mathrm{d}H_X - \mathcal{L}_w D + g_D = \mathrm{d}H_T - \iota_w \mathrm{d}D + g_D, \quad B_t = -\mathrm{d}E_X - \mathcal{L}_w B + g_B = -\mathrm{d}E_T - \iota_w \mathrm{d}B + g_B. \tag{28}$$

Its symbol is $J(\xi)C - (w\cdot\xi)I$, where $J(\xi) = \left(\begin{smallmatrix} 0 & \mathrm{Cross}(\xi) \\ -\mathrm{Cross}(\xi) & 0 \end{smallmatrix}\right)$ is symmetric. Multiplication by C symmetrizes the symbol. Constraints satisfy $c_t = -\mathcal{L}_w c + \mathrm{d}g$. Their characteristics have velocity w, which vanishes in a port collar and points into the hole near the horizon. This choice avoids introducing an incoming constraint mode at the port.

The spatial X-energy flux and its local identity are

$$J_X = E_X \times H_X + w e_X, \qquad \partial_t(e_X \mathrm{d}V_h) + \mathrm{div}_h\, J_X \mathrm{d}V_h = (\mathcal{R}_X(F) + \mathrm{Re}\langle CF, g\rangle)\mathrm{d}V_h. \tag{29}$$

Here $\mathcal{R}_X$ is the quadratic deformation term. The augmented equations have possible constraint currents parallel to X; their contraction with the antisymmetric Maxwell tensor has zero work against X. Thus (29) has no extra constraint-source term. For $r \geq 8M$, $X = T$ is Killing and $\mathcal{R}_X = 0$.

Theorem 4.1 (Physical core input and output)**.** *For $F_0 \in \mathcal{K}$, $g \in L^1(0,T;\mathcal{K})$ and arbitrary incoming $f \in \mathcal{P}$, there is a unique physical weak solution $F \in C([0,T];\mathcal{K})$ with a completed outgoing power trace $y \in \mathcal{P}$. There is a canonical nonnegative additive horizon loss Φ_H such that*

$$E_X(F(t)) + \|y\|^2_{\mathcal{P}(0,t)} + \Phi_H([0,t]) = E_X(F_0) + \|f\|^2_{\mathcal{P}(0,t)} + \int_0^t \int_\Omega (\mathcal{R}_X(F) + \mathrm{Re}\langle CF, g\rangle)\mathrm{d}V_h \mathrm{d}s. \tag{30}$$

If $\mathcal{R}_X \leq 2\omega e_X$ with $\omega \geq 0$, then

$$E_X(F(t)) + \|y\|^2_{\mathcal{P}(0,t)} + \Phi_H([0,t]) \leq 3e^{2\omega t}\big[E_X(F_0) + \|f\|^2_{\mathcal{P}(0,t)} + \|g\|^2_{L^1(0,t;\mathcal{H})}\big]. \tag{31}$$

Both charges are retained and obey their bulk-source flux identities. In particular they are constant for $g = 0$, including for arbitrary incoming radiative histories.

Proof. We give the boundary existence argument because the energy inequality alone would not establish the claim. Continue the coefficients to the annulus $M < r < R$. At $r = M$ the inner characteristic form is strictly outflow for fixed $|a| < M$; call its positive power trace h_b. No datum is imposed there. Let A_X be (28)'s expression. Near the outer boundary $w = 0$, and its graph condition is precisely $E_T, H_T \in H(\mathrm{curl})$. Their tangential traces exist distributionally. Requiring $f, y \in L^2$ is equivalent to requiring those tangential traces in L^2, without a normal trace.

The following density construction justifies Green's formula in that graph/trace space. Flatten a port chart and smooth tangentially. With the tangential smoothing scale fixed, $\partial_z E_A = \partial_A E_z + (\mathrm{d}E)_{zA}$ supplies L^2 normal derivatives of tangential components. Even reflection of tangential and odd reflection of normal components extends an $H(\mathrm{curl})$ form without a tangential jump. Normal smoothing at a smaller scale gives convergence both in the graph norm and in the specified L^2 tangential traces. At the inner boundary, divide by the invertible normal matrix before tangential smoothing; the normal coefficient becomes the identity and the first-order coefficient commutator is bounded on L^2 and tends to zero. Then every component has an L^2 normal derivative. A diagonal smoothing and a finite partition give the required global approximation. Consequently

$$\langle A_X U, V\rangle_{\mathcal{H}} + \langle U, A_X V\rangle_{\mathcal{H}} = 2\langle U, \mathsf{R}V\rangle_{\mathcal{H}} + 2\langle f_U, f_V\rangle - 2\langle y_U, y_V\rangle - 2\langle h_U, h_V\rangle, \tag{32}$$

where R is bounded symmetric and $\langle U, \mathsf{R}U\rangle = \int \mathcal{R}_X(U)$. Its formal adjoint is $A_X^\dagger = -A_X + 2\mathsf{R}$.

Take $\lambda > \omega$ for a finite annular coefficient bound and $a_\lambda = \lambda - \omega$. For smooth tests V, (32) gives

$$\frac{a_\lambda}{2}\|V\|^2 + \|f_V\|^2 \le \frac{1}{2a_\lambda}\|(\lambda - A_X^\dagger)V\|^2 + \|y_V\|^2 + \|h_V\|^2. \tag{33}$$

For any bulk datum H_0 and incoming f_0, the functional $\langle H_0, V\rangle + 2\langle f_0, f_V\rangle$ is therefore bounded on the range of $V \mapsto ((\lambda - A_X^\dagger)V, \sqrt{2}y_V, \sqrt{2}h_V)$. Extend to the closure of that range, by zero on its orthogonal complement, and apply Riesz representation. It yields $(U, \sqrt{2}y, \sqrt{2}h)$ satisfying

$$\langle U, (\lambda - A_X^\dagger)V\rangle + 2\langle y, y_V\rangle + 2\langle h, h_V\rangle = \langle H_0, V\rangle + 2\langle f_0, f_V\rangle. \tag{34}$$

Interior tests give $(\lambda - A_X)U = H_0$; independent boundary tests identify the distributional traces with f_0, y, h. They are now L^2, so the graph density already proved applies to U. Its energy identity gives

$$(\lambda - \omega)\|U\|^2 + \|y\|^2 + \|h\|^2 \le \mathrm{Re}\langle U, H_0\rangle + \|f_0\|^2. \tag{35}$$

This proves existence, uniqueness and the resolvent bounds for the actual weak boundary realization.

The homogeneous domain consists of graph fields with $f = 0$ and $y, h \in L^2$. Compactly supported smooth fields prove density in $\mathcal{H}$; (35) for differences proves closedness, including convergence of the traces. The resolvent is onto. After shifting by ω, the dissipative Yosida resolvents generate bounded exponentials; the resolvent identity makes these converge on the dense resolvent domain, and their uniform contraction bound extends the limit to all of $\mathcal{H}$. This constructs the strongly continuous homogeneous semigroup.

Let $D_\lambda f$ be the resolvent lift with $H_0 = 0$. For time-smooth f vanishing initially set $W = D_\lambda f$ and solve the homogeneous boundary problem for Z with source $g + \lambda W - W'$. Then $F = Z + W$ has the prescribed input. Its energy estimate has no derivative of f:

$$\|F(t)\|^2 + 2\|y\|^2_{L^2(0,t)} + 2\|h\|^2_{L^2(0,t)} \le e^{2\omega t}\big(\|F_0\| + \sqrt{2}\|f\|_{L^2(0,t)} + \|g\|_{L^1(0,t;\mathcal{H})}\big)^2. \tag{36}$$

Approximation therefore completes F in $C_t\mathcal{H}$ and y, h in L^2_t for the final declared data class. Passing the spacetime Green formula identifies these limits as the physical power traces. For uniqueness of any weak solution, its temporal primitive belongs to the homogeneous generator domain for zero data and solves $W' = A_B W$, $W(0) = 0$; hence it vanishes.

Restrict to the physical core. Near its horizon $X = \alpha n$ and $J_X = -\alpha J$ with J from (22). A nondecreasing radial cutoff η_ε defines a horizon loss $\int \eta_\varepsilon' \alpha J \cdot \mathrm{d}r$. Its negative part tends to zero by the collar estimate in Theorem 3.2. Every other term of the localized identity converges, so this loss has a unique nonnegative limit on every time interval, giving (30). This also proves independence of the interior extension. Local weak identities follow by time convolution, then the graph/trace approximation at fixed radial cutoff; no original pointwise trace is assumed. The scalar energy comparison and a three-term square inequality give (31) with the physical-core coefficient.

Finally $c_t + \mathcal{L}_w c = 0$ on the core for compatible data. Backward characteristics remain in the core: w is inward at the horizon and zero near the port. Pulling compact tests along that flow proves $c = 0$. Thus the augmented solution is physical. Pairing with $\mathrm{d}\chi_H$, for a cutoff zero near the port, gives the charge identities exactly as in (24). □

5 A verified signed coefficient

From this point onward all coordinates and times are dimensionless in units of M, so $M = 1$ and $a = 0.9$ unless a factor is restored explicitly. A coefficient quoted in these units must be divided by M in dimensional equations.

Use Cartesian densitized flux components $Z = (D_{23}, D_{31}, D_{12}, B_{23}, B_{31}, B_{12})$. The energy is $E_X = \frac{1}{2}\int Z^*\mathsf{C}Z\mathrm{d}^3x$, with

$$\mathsf{C} = \begin{pmatrix} Q & \mathrm{Cross}(b) \\ -\mathrm{Cross}(b) & Q \end{pmatrix}, \quad Q = \frac{I + 2H_K\ell\ell^T}{1 + 2H_K}, \qquad b = (1-\chi)\beta. \tag{37}$$

This Cartesian matrix incorporates the volume conversion from (12); it is distinct from writing the orthonormal matrix C with measure $\mathrm{d}V_h$. It satisfies $\mathsf{C} \geq I/3$. If $\mathsf{D} = \mathrm{diag}(\partial_j w^i, \partial_j w^i)$, direct integration by parts in (28) gives

$$\mathsf{K} = \tfrac{1}{2}(\mathsf{C}\mathsf{D} + \mathsf{D}^T\mathsf{C} - (\mathrm{div}\, w)\mathsf{C} + w^j\partial_j\mathsf{C}), \qquad \mathcal{R}_X\mathrm{d}V_h = Z^*\mathsf{K}Z\mathrm{d}^3x. \tag{38}$$

In particular $\mathsf{K} = 0$ for $r \geq 8$. Define

$$\omega_* = \max\left(0, \sup_{r_+<r<8,\, -1\leq\mu\leq 1} \lambda_{\max}(\mathsf{K}, \mathsf{C})\right), \quad \mu = z/r. \tag{39}$$

Then $\mathcal{R}_X \leq 2\omega_* e_X$. This is a pointwise coefficient, not the observed growth rate of a solution.

Proposition 5.1 (Signed enclosure)**.** *For the exact multiplier* (8)*,*

$$0.499180315738376 \;\leq\; M\omega_* \;\leq\; 0.525452399065317. \tag{40}$$

The sharper upper value used in the numerical estimates is the exact rational

$$U = 0.525452399065316647288330943016181365664305031970977783203125. \tag{41}$$

Verification method. Axisymmetry reduces the coefficient calculation to (r, μ) while retaining Cartesian field components and all their derivatives, including azimuthal derivatives. Cover the closed rectangle from a downward enclosure of $r_+ = 1 + \sqrt{19/100}$ to 8 and $-1 \leq \mu \leq 1$ by exact interval boxes. On each box, enclose (38) using outward arithmetic and certify $U\mathsf{C} - \mathsf{K} \succeq 0$ by interval LDL^T pivots, or by the retained safe coefficient bound where applicable. Positive pivots for an enclosing box also certify every child box. Analytic bounds for $s^{-k}e^{-1/s}$ handle the flat endpoints of χ; the exterior $r \geq 8$ is handled analytically.

The finite certificate consists of 49,794 covering leaves and 47,810 matrix proof objects. Its endpoint and subdivision checks establish a complete partition, including axes and horizon. Separately, at 1,471 physical points above the horizon, exact rational vectors give outward Rayleigh-quotient lower enclosures. The largest certified lower endpoint is

$$0.499180315738376015373224354244408308132 9.$$

The upper and lower comparisons use the positive denominator $Z^*\mathsf{C}Z$; sampled eigenvalues do not establish either inequality. Replaying all proof objects and physical lower witnesses gives (40). The public supplement supplies these finite numerical witnesses and the evaluators, so the numerical portion of the proposition is a reproducible interval certificate. □

The signed upper bound improves the preceding Frobenius-norm estimate by a factor of approximately 9.53. The positive lower bound prevents making this particular pointwise coefficient arbitrarily small by tighter arithmetic. It neither proves an unstable physical mode nor bounds a different energy construction.

6 The $80M$ causal-region error estimate

A global application of (31) uses the entire observation duration in its exponential. We now restrict the duration of its deformation term using the causal energy current. No geometric-optics approximation or trapping-time assumption is needed.

For $r \geq 8$ define

$$v_{\mathrm{in}}(r) = \frac{\sqrt{r^4 + a^2 r(r+2)} + 2r}{r^2 + 2r}, \qquad v_{\mathrm{out}}(r) = \frac{\sqrt{r^4 + a^2 r(r+2)} - 2r}{r^2 + 2r}. \tag{42}$$

These are positive sphere maxima of the magnitudes of the inward and outward coordinate radial causal speeds, respectively. To check the maxima, substitute $|\mathrm{d}r|_h^2 = (r^2+a^2)/(r^2 + a^2\mu^2) - 2H_K/(1+2H_K)$ into $-\beta^r \pm \alpha|\mathrm{d}r|_h$; the extrema of the corresponding inward and outward magnitudes occur at $\mu = 0$. More explicitly, put $s = r^2 + a^2\mu^2$, $A_r = r^2 + a^2$ and $\Delta = A_r - 2r$. The magnitudes are $(\sqrt{\Delta s + 2rA_r} \pm 2r)/(s+2r)$. Their derivatives in s have numerators proportional to $-\Delta s - 2r(A_r + 2r) \mp 4r\sqrt{\Delta s + 2rA_r}$. The inward one is negative. For the outward one, $s \leq A_r$ and $\sqrt{\Delta s + 2rA_r} \leq A_r$ bound that numerator by $\Delta(2r - s) < 0$. Both maxima thus occur at the smallest $s = r^2$. The causal Maxwell current has coordinate ratio J_X/e_X, and therefore

$$-v_{\mathrm{in}} e_X \leq J_X \cdot \mathrm{d}r \leq v_{\mathrm{out}} e_X. \tag{43}$$

Indeed in an orthonormal frame with time axis parallel to X, its energy-momentum density is a positive multiple of $(e, u \times v)$, and $|u \times v| \leq e$. Future causality is invariant under a change of frame, including in the presence of sources.

Set

$$a_-(r) = \int_r^R \frac{\mathrm{d}s}{v_{\mathrm{in}}(s)}, \qquad b_+(r) = \int_r^R \frac{\mathrm{d}s}{v_{\mathrm{out}}(s)} \quad (8 \leq r \leq R), \tag{44}$$

and clamp them to $A = a_-(8)$ and $B = b_+(8)$ below 8. They are continuous Lipschitz functions. Certified lower travel endpoints are the following exact finite decimals:

$$a_0 = 991.830171577935420800049281825100525355 9 \leq A, \tag{45}$$

$$b_{\mathrm{low}} = 1012.240831910646907845561488451548166895 \leq B. \tag{46}$$

Fix, exactly,

$$\begin{gathered} T_1 = a_0 + b_{\mathrm{low}} + 80 = 2084.071003488582328645610770276648692250 9, \\ T_0 = T_1 - 80, \qquad I_{\mathrm{obs}} = [T_0, T_1]. \end{gathered} \tag{47}$$

These are port times in the stationary Kerr–Schild clock t. They are not identified with retarded incoming or outgoing clocks at null infinity.

Theorem 6.1 (Regional estimate and compatible remainder)**.** *Let an error field have zero initial data, incoming error $f_{\mathrm{good}} \in \mathcal{P}$ supported in $[0, 40]$, and a compatible source $g_{\mathrm{good}} \in L^1(0, T_1; \mathcal{K})$ such that*

$$g_{\mathrm{good}}(t, x) = 0 \quad \textit{for almost every } (t, x) \textit{ with } t < a_-(r). \tag{48}$$

Then its actual outgoing trace obeys

$$\|y_{\mathrm{good}}\|^2_{\mathcal{P}(0,T_1)} \leq B_{\mathrm{good}} := 3e^{160U}(\|f_{\mathrm{good}}\|^2_{\mathcal{P}} + \|g_{\mathrm{good}}\|^2_{L^1_t \mathcal{H}}). \tag{49}$$

For a general error, suppose an exact sum of separately compatible problems is supplied, with all initial data F_{bad} and remaining incoming and bulk errors assigned to the second problem. Put

$$B_{\mathrm{bad}} = 3e^{2UT_1}(E_X(F_{\mathrm{bad}}) + \|f_{\mathrm{bad}}\|^2_{\mathcal{P}} + \|g_{\mathrm{bad}}\|^2_{L^1_t \mathcal{H}}). \tag{50}$$

Then on the unchanged I_{obs},

$$\|\delta y\| \leq \varepsilon_y := \sqrt{B_{\mathrm{good}}} + \sqrt{B_{\mathrm{bad}}}, \quad \|\delta y\|^2 \leq (\sqrt{B_{\mathrm{good}}} + \sqrt{B_{\mathrm{bad}}})^2. \tag{51}$$

Without a certified compatible split, the whole error must use the global bound.

Proof. For a Lipschitz weight q, $0 \le q \le 1$, spatially constant in a horizon collar, the weak identity is

$$E_q(t) + \int_0^t q_R(s)(\|y(s)\|^2_{\mathrm{port}} - \|f(s)\|^2_{\mathrm{port}})\mathrm{d}s + \int_{[0,t]} q_H \mathrm{d}\Phi_H = E_q(0) + \int_0^t \int \big[q\mathcal{R}_X + q\,\mathrm{Re}\langle Cz, g\rangle + q_t e_X + J_X \cdot \mathrm{d}q\big]\mathrm{d}V_h \mathrm{d}s. \tag{52}$$

Here $E_q = \int q e_X \mathrm{d}V_h$. Time convolution and the graph identity establish it for smooth weights. Smooth approximations of Lipschitz radial weights have uniformly bounded derivatives and almost-everywhere derivative convergence. Dominated convergence passes the L^1 volume terms, strong $C_t\mathcal{H}$ continuity passes the slices, and the actual L^2 traces pass the port powers. In the constant horizon collar the same horizon cutoff gives exactly $q_H \mathrm{d}\Phi_H$. The kink at $r = 8$ adds no measure because the weight itself is continuous.

Let θ be smooth nondecreasing, zero on $(-\infty, 0]$ and one on $[1, \infty)$. With $q^-_\varepsilon = \theta((a_-(r)-t)/\varepsilon)$, the derivative term outside 8 is $-(\theta'/\varepsilon)(e_X + J_X \cdot \mathrm{d}r/v_{\mathrm{in}}) \le 0$; inside it is $-(\theta'/\varepsilon)e_X$. The source term vanishes by (48), the outer weight is zero, and the initial weighted energy vanishes. Gronwall gives $E_{q^-_\varepsilon} = 0$, hence

$$z(t, x) = 0 \quad \text{almost everywhere where } t < a_-(r). \tag{53}$$

This proof does not approximate the source by an unproved support-preserving projection.

Next use $q^+_\varepsilon = \theta((T_1 - t - b_+(r))/\varepsilon)$. Its derivative contributes the nonnegative loss

$$\mathrm{d}\Lambda_\varepsilon = -\big[(q^+_\varepsilon)_t e_X + J_X \cdot \mathrm{d}q^+_\varepsilon\big]\mathrm{d}V_h \mathrm{d}t \ge 0. \tag{54}$$

Outside 8 this follows from $e_X - J_X \cdot \mathrm{d}r/v_{\mathrm{out}} \ge 0$; inside it is a nonnegative multiple of e_X. On $r < 8$, (53) excludes $t < a_0$, whereas $q^+_\varepsilon > 0$ implies $t < T_1 - B \le a_0 + 80$. Since deformation vanishes for $r \ge 8$, its entire weighted contribution is bounded by $2c(t)E_{q^+_\varepsilon}$ with

$$c(t) = U\mathbf{1}_{[a_0, a_0+80]}(t), \qquad \int_0^{T_1} c(t)\mathrm{d}t = 80U. \tag{55}$$

This bounds deformation exposure in the containing region, irrespective of trapping.

Add to the weighted slice energy the accumulated outgoing power, horizon loss, and (54), obtaining a nonnegative quantity $A_\varepsilon(t)$. The balance is absolutely continuous as a sum even when individual loss terms are measures, and

$$A'_\varepsilon \le 2cA_\varepsilon + \|f\|^2_{\mathrm{port}} + \sqrt{2A_\varepsilon}\|g\|_{\mathcal{H}}, \qquad A_\varepsilon(0) = 0. \tag{56}$$

The source bound is $|\int q\langle Cz, g\rangle| \le \sqrt{2E_q}\|g\|_{\mathcal{H}}$, needing no derivative or trace of g. Multiplying by $e^{-2\int c}$ and comparing with $(F + G)^2$, where $F(t)^2 = \int_0^t e^{-2\int_0^s c}\|f(s)\|^2\mathrm{d}s$ and $G(t) = 2^{-1/2}\int_0^t e^{-\int_0^s c}\|g(s)\|\mathrm{d}s$, gives

$$A_\varepsilon(t) \le e^{2\int_0^t c}\big(\|f\|_{L^2(0,t)} + 2^{-1/2}\|g\|_{L^1(0,t;\mathcal{H})}\big)^2. \tag{57}$$

A positive initial regularization justifies comparison at $F = 0$. The weaker factor 3 in (49) safely bounds the square. At T_1 the slice weight vanishes and $q^+_R \to 1$ for every $t < T_1$. Dominated convergence of the actual outgoing trace proves (49), indeed on all of $[0, T_1]$. Restriction to I_{obs} is contractive. Linearity, uniqueness and the triangle inequality prove (51). □

6.1 The weak cap and the error square

The cap losses (54) have a uniform finite bound. Their weak measure limit is unique, not a choice of trace or subsequence: in (52), the weights tend to $\mathbf{1}_D$ for $D = \{t + b_+(r) < T_1\}$, the volume terms converge in L^1, and the slice energies converge in L^1 time and on each slice. The horizon loss has no time atoms because (30) expresses its cumulative value by continuous slice energy and integrals of L^1 functions. Its weighted limit is therefore its restriction to $t < T_1 - B$. These limits determine a unique nonnegative measure Λ_D. With a right-continuous convention at the clamped interior cap,

$$\begin{aligned} &E_D(t) + \|y\|^2_{\mathcal{P}(0,t)} + \Phi_H([0,t] \cap \{s < T_1 - B\}) + \Lambda_D([0,t]) \\ &\quad = \|f\|^2_{\mathcal{P}(0,t)} + \int_0^t \int_D (\mathcal{R}_X + \mathrm{Re}\langle Cz, g\rangle) \mathrm{d}V_h \mathrm{d}s. \end{aligned} \tag{58}$$

The jump removing $r < 8$ at $t = T_1 - B$ contributes a nonnegative spacelike-cap energy atom. It is included in Λ_D. At T_1 the region has no spatial volume, and strong continuity makes the shrinking outer-annulus energy tend to zero. Source outside D was not cut into a different Maxwell problem: (58) is a weighted identity for the original solution. Likewise a causal indicator applied to a conserved source does not generally preserve compatibility, so it cannot automatically produce the decomposition in Theorem 6.1.

For any certified reference output $\widehat{y}$ on I_{obs}, the resulting physical power enclosure is

$$\max(0, \|\widehat{y}\| - \varepsilon_y)^2 \le \|y\|^2 \le \|\widehat{y}\|^2 + 2\|\widehat{y}\|\varepsilon_y + \varepsilon_y^2. \tag{59}$$

Both the positive squared error and the cross term between the two error classes remain. Initial errors, including arbitrary charge errors, use the global part unless a separate valid estimate is supplied.

For a pure good-class error, a sufficient condition for $\varepsilon_y \le 10^{-3}$ is

$$(\|f\|^2_{\mathcal{P}} + \|g\|^2_{L^1_t \mathcal{H}})^{1/2} \le \frac{10^{-3}}{\sqrt{3} e^{80U}} \simeq 3.20149759039437 \times 10^{-22}. \tag{60}$$

This is a residual-norm requirement of a sufficient certificate, not a statement about required arithmetic digits or actual physical error.

7 A closed-potential reference for a fixed pulse

The physical initial state and bulk source in the example are exactly zero. Define the regular $m = 1$ scalar $Y = (x + iy)/(r + ia)$ and its port tangential differential $Z_1 = \mathrm{d}_S Y|_{r=R}$. The incoming field is fixed before approximation:

$$\begin{aligned} f(t) = A_p p(t) Z_1, \qquad p(t) &= \begin{cases} \sin^2(\pi t/40) e^{-0.15it}, & 0 < t < 40, \\ 0, & \text{otherwise}, \end{cases} \\ A_p = (15 N_Z)^{-1/2}, \qquad N_Z &= \int_{r=R} |Z_1|^2_\gamma \mathrm{d}A_\gamma. \end{aligned} \tag{61}$$

The envelope is C^1; $\int_0^{40} |p|^2 \mathrm{d}t = 15$, so $\|f\|_{\mathcal{P}} = 1$. This is one electric-parity complex-normalized port pulse, not an arbitrary-input operator calculation.

Write the reference as $\widehat{D} = \mathrm{d}P$, $\widehat{B} = \mathrm{d}Q$ (Cartesian curls in densitized vector notation). Let

$$\tau = t - (R - r), \qquad \eta = \mathrm{d}x + i\mathrm{d}y. \tag{62}$$

Base potentials are linear combinations of $v\eta$, $vY\mathrm{d}z$, and $vY\mathrm{d}r$, independently for P and Q. Scalar v ranges over products of open cubic radial B-splines, open cubic time B-splines in τ,

and $1, \mu$. Radial breaks are $r_+, 8, 64, 1000$; time breaks are $0, 40, 120, 600, 2100$. End knots have multiplicity four. Only time basis indices $3, 4, 5, 6$ (zero-based) are used, ensuring value and first two derivatives vanish at $\tau = 0$. There are $6 \cdot 6 \cdot 4 \cdot 2 = 288$ complex coefficients. The prescribed refinement inserts one midpoint in each time span; its first span is $0 < \tau < 20$. The obstruction below applies regardless of additional spatial coefficients, provided the front, correction and cubic first-span time dependence remain as specified.

7.1 Exact incoming correction with unchanged base outgoing trace

At the port let $n_r = \nabla_{\mathbb{R}^3} r$, $N = |n_r|$, $\nu = n_r/N$, and $\Pi = I - \nu\nu^T$. The physical stationary constitutive relations for Cartesian flux vectors are

$$E_T = QD + \beta \times B, \qquad H_T = QB - \beta \times D. \tag{63}$$

A potential correction vanishing at the port with radial jet u_P changes D by $n_r \times u_P$. Every tangent increment d is obtainable by $u_P = -\nu \times d/N$. For tangent increments the constitutive matrix, extended by normal identities, is

$$C_t = \begin{pmatrix} A_t & \beta_n \,\mathrm{Cross}(\nu) \\ -\beta_n \,\mathrm{Cross}(\nu) & A_t \end{pmatrix}, \quad A_t = q_0 \Pi + f_K \ell_t \ell_t^T + \nu\nu^T, \tag{64}$$

where $q_0 = (1 + 2H_K)^{-1}$, $f_K = 2H_K/(1 + 2H_K)$, $\ell_t = \Pi\ell$, and $\beta_n = \nu \cdot \beta$. Its tangent quadratic form is bounded below by $(q_0 - |\beta_n|)(|d|^2 + |b|^2)$, and at R this is at least $(R-2)/(R+2) = 998/1002$ times that norm. Thus the inverse exists uniformly over the entire port, including the axes.

For a desired incoming correction $w_f = f - f_{\mathrm{base}}$ prescribe $\delta E_t = w_f$ and $\delta H_t = -J_\gamma w_f$. Then $\delta f = w_f$ and $\delta y = 0$. Solve (64) and recover the two radial potential jets. Extend them into a collar by

$$P_{\mathrm{corr}} = (r - R)\chi_{\mathrm{col}}(r) u_P(\tau, \mathrm{angles}), \qquad Q_{\mathrm{corr}} = (r - R)\chi_{\mathrm{col}}(r) u_Q(\tau, \mathrm{angles}),$$
$$\chi_{\mathrm{col}} = \left(\frac{(r - R + 8)_+}{8} \right)^3. \tag{65}$$

Angular transport in these formulas keeps μ and Y fixed when evaluating the port jets. Explicitly the port point has $x_p + iy_p = (R + ia)Y$, $z_p = R\mu$. The formulas are regular in Cartesian components and equivariant under axial rotations. At the port (65) has exactly the specified radial jet, hence the incoming trace equals the fixed (61) for every time, including after the pulse, and the outgoing trace equals the base outgoing trace.

Since $v_{\mathrm{in}} > 1$ on $[8, R]$, $a_-(r) < R - r$ there, and the clamp below 8 also gives $a_-(r) \le R - r$. The reference and its residual vanish for $\tau < 0$, so they satisfy the forward support condition. The C^2 spline joins and collar join, together with the C^1 pulse and its bounded piecewise second derivative, give potential weak second derivatives locally in L^2. There are no residual delta functions at those joins. The full compatible error source is

$$g_D = \mathrm{d}(H_T - \partial_t P), \qquad g_B = -\mathrm{d}(E_T + \partial_t Q). \tag{66}$$

Thus zero initial error, zero incoming mismatch, and the good-class source conditions are exact properties of this construction. No incompatible source cutoff is used. The physical bulk forcing is still zero; (66) is approximation error.

7.2 Full residual enclosure and its limited numerical outcome

A least-squares fit of 288 complex coefficients on 9,216 guidance rows reduced the sampled spacetime residual norm from approximately 1.09162 to 1.05364. The fit only selects the reference. All coefficients were then frozen as exact dyadic rationals; the exact algebraic horizon

knot and exact normalization A_p are retained in its mathematical definition. The complete residual is evaluated independently of fitting nodes.

The enclosure uses mixed potential derivatives through total order three, which are necessary for second derivatives of the retarded port correction. It encloses residual values, not residual derivatives inferred from truncated derivative arrays. The cubic recurrence

$$N'_{i,p} = \frac{pN_{i,p-1}}{k_{i+p} - k_i} - \frac{pN_{i+1,p-1}}{k_{i+p+1} - k_{i+1}} \tag{67}$$

with zero terms for repeated knots supplies interval derivatives. Boxes crossing joins enclose all piecewise values; weak matching already excludes missing delta contributions. All geometry is differentiated at fixed physical t, so spatial derivatives include the $\tau = t - R + r$ chain rule. Cartesian jets, rather than derivatives of polar square roots, handle axes.

The normalization is enclosed analytically. With $k = 2/(R-2)$,

$$\frac{16\pi/3}{(1 + a^2/R^2)(1+k)} \le N_Z \le \frac{16\pi}{3}(1 + a^2/R^2)\sqrt{1+k}. \tag{68}$$

This follows from the oblate-sphere metric relative to the round metric and the stationary screen correction; $\int_{S^2} |\mathrm{d}_S Y|^2 = 16\pi/3$. The corresponding fixed A_p lies between 0.06304672467805 and 0.06314151171130. The same interval is propagated through every cell; the input is not reselected cell by cell.

A 160-cell residual cover includes the complete radial, angular, and time domain. A 16-cell output cover includes both exact observation endpoints, rounded outward. The fields are pure $m = 1$ tensors, and rotation equivariance makes complex energy independent of azimuth in a rotated Cartesian frame. Thus the full azimuthal integral is exactly 2π times the meridional value; no angular residual tail is discarded. The Cartesian Jacobian is $r^2 + a^2\mu^2$, and $\mathsf{C} \le 2I$ gives a safe energy upper bound. The moving time cap $\tau \le T_1 - R + r$ is safely overcovered. A final Cauchy–Schwarz estimate gives

$$\|g\|^2_{L^1_t \mathcal{H}} \le T_1 \int_0^{T_1} \|g(t)\|^2_{\mathcal{H}} \mathrm{d}t. \tag{69}$$

The spacetime energy-integral upper bound is approximately 1.2702871×10^{45}. The resulting source-norm-square upper bound is 2.6473686×10^{48} and output-amplitude error upper bound is 5.0822266×10^{42}. Equation (59) therefore yields a physical output-power enclosure from zero to approximately 2.5829027×10^{85}. This complete enclosure supplies no useful positive lower bound on core return. A nonzero fitted field or sampled outgoing value does not change that conclusion.

8 An exact obstruction for the two cubic families

The wide interval bound alone would leave open whether better fitting or quadrature could meet (60). The next result excludes that possibility for the two stated cubic time families while retaining the fixed input, front and port correction.

Theorem 8.1 (Coefficient-independent sufficient-norm obstruction)**.** *For every complex coefficient tensor in the reference family of Section 7, its full physical residual satisfies*

$$\|g\|_{L^1(0,T_1;\mathcal{H})} \ge 1.07200524042456 \times 10^{-5}. \tag{70}$$

For every coefficient tensor of its prescribed one-level midpoint time refinement, with the same front and collar correction,

$$\|g\|_{L^1(0,T_1;\mathcal{H})} \ge 1.32991237024562 \times 10^{-6}. \tag{71}$$

Consequently neither family can satisfy the sufficient criterion (60) by coefficient tuning, increased arithmetic precision, or tighter residual quadrature. These are residual lower bounds, not lower bounds on true output error.

Proof. On the first time span every base potential is a polynomial of degree at most three in τ: the span is $h = 40$ for the original family and $h = 20$ for the refinement. The geometry, physical differential operator and port inverse are stationary. Differentiation and multiplication by stationary coefficients do not increase time degree. The coefficient-dependent port correction evaluates the base port trace at the same τ. Therefore every coefficient-dependent component of the *full* residual, including that correction and its mixed derivatives, has time degree at most three on this span.

Let

$$W_h(\tau) = P_4(2\tau/h - 1), \qquad P_4(s) = \tfrac{1}{8}(35s^4 - 30s^2 + 3). \tag{72}$$

It has $|P_4| \le 1$ on $[-1, 1]$, and direct integration gives

$$\int_0^1 P_4(2s-1)s^n \mathrm{d}s = \begin{cases} 0, & n < 4, \\ \dfrac{n(n-1)(n-2)(n-3)}{(n+1)(n+2)(n+3)(n+4)(n+5)}, & n \ge 4. \end{cases} \tag{73}$$

Writing $g = g_{\text{fixed}} + \sum_j c_j g_j$, it follows pointwise in space that

$$m(x) := \int_0^h W_h(\tau) g(\tau + R - r, x) \mathrm{d}\tau = \int_0^h W_h(\tau) g_{\text{fixed}}(\tau + R - r, x) \mathrm{d}\tau. \tag{74}$$

Thus the moment annihilates all variable coefficients, not just the computed fit.

The fixed column is linear in p, p', p''. Write

$$p(\tau) = \tfrac{1}{2}e^{\lambda_0 \tau} - \tfrac{1}{4}e^{\lambda_+ \tau} - \tfrac{1}{4}e^{\lambda_- \tau}, \quad \lambda_0 = -0.15i, \quad \lambda_+ = i(\pi/20 - 0.15), \quad \lambda_- = -i(\pi/20 + 0.15). \tag{75}$$

The weighted moments of $p^{(j)}$ follow by multiplying each exponential moment by λ^j. Expand through order 120, integrate by (73), and use the remainder bound

$$h e^{13} \frac{13^{121}}{121!}, \qquad |\lambda| h < 13. \tag{76}$$

Outward rational bounds for π, elementary exponentials and the normalization (68) make this an exact interval operation, not oscillatory floating-point quadrature. Substituting these three moments into the expanded residual value expression computes (74); its spatial derivatives already include the moving-window chain rule.

Use the positive-volume collar box

$$\mathcal{B}: \quad 995.999 \le r \le 996.001, \qquad 0.1999 \le \mu \le 0.2001, \qquad 0 \le \phi < 2\pi. \tag{77}$$

The associated physical times $R - r \le t \le R - r + h$ lie strictly inside $[0, T_1]$. The interval moment calculation identifies a signed real or imaginary component of m bounded below by $m_{\text{low}} > 0$ throughout this box. Since the residual is an $m = 1$ tensor, rotate and phase-adjust its component dual vector with ϕ. The resulting test vector has Euclidean magnitude at most one after multiplication by $W_h(t - R + r)$ and the box/time indicators, and its pairing has the same sign at every azimuth. It is only a dual test function; it is not asserted to be a compatible Maxwell source.

Set $\Delta r = 0.002$, $\Delta\mu = 0.0002$ and

$$V_- = 2\pi r_{\text{lo}}^2 \Delta r \Delta\mu, \qquad V_+ = 2\pi(r_{\text{hi}}^2 + a^2)\Delta r \Delta\mu. \tag{78}$$

Because $\mathsf{C} \ge I/3$, the spatial dual norm of the test is at most $\sqrt{3V_+}$ at each physical time. Fubini (with $\mathrm{d}t = \mathrm{d}\tau$), Cauchy–Schwarz in space and integration of the actual L^1 time norm give

$$\|g\|_{L^1_t \mathcal{H}} \ge \frac{m_{\text{low}} V_-}{\sqrt{3V_+}}. \tag{79}$$

The complete outward evaluation gives respectively the lower endpoints

$$0.000010720052404245641509991370130572450086 88,$$
$$0.000001329912370245624461110326505259586794165.$$

Rounding down yields the theorem. Both exceed (60) by many orders of magnitude. □

The positive volume, all-azimuth pairing and moving physical-time support are essential. A nonzero point residual would not by itself prove (79). Conversely the present theorem does not cover higher time degree, further subdivisions, a different support front or collar, a pulse-adapted time basis, or a cancellation-sensitive source-to-output estimate. It does not show that these two families have output errors larger than 10^{-3}; it shows that the selected sufficient norm test cannot certify that tolerance for them.

9 Interpretation and limitations

The analytic results construct a physical domain, compatible forcing and a weak power observation, retaining both charge sectors. Their finite constants do not constitute useful long-time stability or response estimates. In particular a positive signed energy-production coefficient is not a physical instability, and its upper bound is not an observed growth rate.

The regional estimate is stronger than applying the global exponential over the full port interval, but its conditions are substantive. It requires zero initial error, the stated incoming support, and an actual forward-supported compatible bulk residual. Nonzero initial data and unsupported remainders must be included through the full-duration estimate and combined at amplitude level. The $80M$ duration belongs to the deformation term inside the selected causal region. It is neither a universal residence time nor a statement that trapped waves leave the core in $80M$.

The numerical example proves compatibility and complete residual coverage for a nonzero fixed-input reference. It certifies no useful physical return, no net extracted energy, and no exclusion of a proposed physical configuration. Its moment theorem is an exact limitation of two cubic families under one sufficient certificate. Another reference or a cancellation-sensitive estimate could escape it, but no such successful construction is supplied here.

Several physical identifications also remain separate: general material or interface currents, metric and magnetic-constraint residuals, exterior feedback and repeated propagation, and a norm-controlled correspondence between the finite-port pulse/window and incoming or outgoing null-infinity radiation clocks. The compact-interior current theorem and its optional stronger port-trace assumptions do not establish those identifications automatically. A single electric-parity pulse is not a certified arbitrary-input response operator.

10 Conclusion

Finite-energy Maxwell evolution and a physical finite-port error observation can be constructed on Kerr with explicit source and charge bookkeeping. For the specified positive energy, signed interval bounds and a weak causal-region argument provide a rigorous $80M$ deformation-exposure estimate for a precisely defined error class. Applying that estimate to an actual closed-potential reference gives no useful pulse-response bound. An exact Legendre moment explains one reason further fitting cannot suffice: both declared cubic families have residual floors far above this certificate's sufficient target. The established endpoint is therefore a set of domain and error-bound results together with a restricted approximation obstruction, while useful physical response certification remains open.

Reproducibility and usage

The accompanying reproducibility supplement contains the coefficient cover and witnesses, the exact reference coefficients, complete residual and output certificates, moment evaluation, and verification scripts. Fit samples guide coefficient selection only; acceptance of numerical inequalities uses the full outward certificates. The interval checks include the axes, physical horizon and exact observation endpoints. The manuscript's analytic proofs specify the domains and signs on which those computations depend. Text and manuscript source are licensed under CC BY 4.0; original supplement software and accompanying documentation use the MIT licence. This document is a preprint.

A Explicit lift constants and differential-domain control

This appendix supplies the geometry estimates used in Proposition 3.1. For $r \geq r_+ \geq M$, $I \leq h \leq 3I$ and $1 \leq \sqrt{\det h} \leq \sqrt{3}$. In oblate coordinates $x = \sqrt{r^2+a^2}\sin\theta\cos\phi$, $y = \sqrt{r^2+a^2}\sin\theta\sin\phi$, $z = r\cos\theta$, the radial speed satisfies $1/\sqrt{2} \leq |\partial_r x|_{\mathbb{R}^3} \leq 1$ and $|\partial_r|_h^2 \leq 3$. The Euclidean area element is $\sqrt{r^2+a^2}\sqrt{r^2+a^2\cos^2\theta}\sin\theta \mathrm{d}\theta\mathrm{d}\phi$, increasing with r. Thus on $r_+ \leq r \leq r_+ + \delta$, $\mathrm{d}V_h \geq (3\sqrt{2})^{-1}\mathrm{d}A_h(r_+)\mathrm{d}r$. Parallel transport along radial curves and the one-dimensional fundamental theorem give

$$\|v|_{r=r_+}\|_2^2 \leq a_\delta \|v\|_{2,\mathrm{col}}^2 + b_\delta \|\nabla v\|_{2,\mathrm{col}}^2, \quad a_\delta = 6\sqrt{2}/\delta, \quad b_\delta = 18\sqrt{2}\delta. \tag{80}$$

This is an H^1 trace estimate used only for the auxiliary elliptic lift.

Let $R_- = \sup \max(0, -\lambda_{\min}(\mathrm{Ric}_h))$ and $k_- = \sup \max(0, -H_\Gamma)$, where H_Γ uses the normal pointing out of the exterior toward the hole. These are finite explicit geometry coefficients obtained from the displayed metric and its first two derivatives. Take $\delta = M$ if $k_- = 0$, and otherwise $\delta = \min(M, (36\sqrt{2}k_-)^{-1})$. Then $k_- b_\delta \leq 1/2$ and

$$C_{\mathrm{geo}} = 2R_- + 2k_- a_\delta, \qquad \|\nabla^2\phi_q\|_2^2 \leq 2\|\rho\|_2^2 + C_{\mathrm{geo}}\|\mathrm{d}\phi_q\|_2^2. \tag{81}$$

To prove it, solve the Dirichlet problem first on a truncated exterior with a sufficiently large, mean-convex outer sphere. The integrated Bochner identity is

$$\int (\Delta\phi)^2 = \int |\nabla^2\phi|^2 + \int \mathrm{Ric}(\mathrm{d}\phi, \mathrm{d}\phi) + \int_{\partial\Omega} H_{\partial\Omega}(\partial_n\phi)^2. \tag{82}$$

The outer term is favorable; the inner one is controlled by (80). Absorbing $k_- b_\delta$ proves (81), uniformly in the outer radius. On bounded domains, tangential difference quotients in boundary charts and the elliptic equation for the normal second derivative justify this identity for weak solutions. Their finite-domain regularity constants do not enter (81). As the outer radius increases, the Dirichlet solutions are Riesz projections onto nested subspaces dense in V. Their gradients converge strongly, while the uniform Hessian bound passes by weak compactness and local distributional testing. Compact L^2 source approximation finishes the argument.

Set

$$\begin{gathered} B_0 = \sup|\beta|_h, \quad B_1 = \sup\|\nabla\beta\|_{\mathrm{op,h}}, \quad A_1 = \sup|\mathrm{d}\alpha|_h, \\ C_\rho = (1+\sqrt{2})B_0, \quad C_q = B_0\sqrt{C_{\mathrm{geo}}} + 2B_1 + A_1. \end{gathered} \tag{83}$$

These are finite coefficient suprema, not response norms. Here $B_0 \leq 2/\sqrt{3}$. Since $D_q = -*\,\mathrm{d}\phi_q$,

$$\begin{gathered} \mathrm{d}\iota_\beta D_q = \mathcal{L}_\beta D_q - \iota_\beta q, \qquad -\mathrm{d}(\alpha * D_q) = -\mathrm{d}\alpha \wedge *D_q, \\ |\mathcal{L}_\beta D_q| \leq |\beta||\nabla D_q| + 2\|\nabla\beta\|_{\mathrm{op}}|D_q|. \end{gathered} \tag{84}$$

The last identity cancels the otherwise problematic derivative. Together with (81), it gives $\|\mathcal{A}(Lq, 0)\|_{\mathcal{H}_0} \leq C_\rho\|\rho\|_2 + C_q\|q\|_{V^*}$, proving maximal differential-domain membership and (17).

For support inside Euclidean radius R_s, the dual norm can itself be bounded explicitly:

$$\|q\|_{V^*} \leq 2\sqrt{3}R_s\|\rho\|_2. \tag{85}$$

Extend a test $\phi \in V$ by zero into the hole and apply $\int |\phi|^2/|x|^2 \leq 4\int |\nabla\phi|^2$, obtained by expanding $\int |\nabla\phi + x\phi/(2|x|^2)|^2 \geq 0$. Metric comparison gives the displayed constant. These constants establish finite source control without an unspecified elliptic inverse or a long-time Maxwell estimate.

References

[1] G. Benomio and R. Teixeira da Costa, *The Maxwell equations on full sub-extremal and extremal Kerr spacetimes*, arXiv:2512.08917v1 (2025), Theorem 5.1 and Sections 3–4. https://arxiv.org/abs/2512.08917.